Size ratio universality in chaotic systems

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In this paper we attempt to predict the size of the chaotic attractor at the parameter value where the period tends to infinity for nonlinear systems that follow the period doubling route to chaos, and for which the Feigenbaum universalities hold.

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1. Introduction

Feigenbaum discovered a pattern in the period doubling route to chaos for nonlinear systems whose first return map is unimodal with a quadratic extremum. This led to the definition of the Feigenbaum universal constant $\delta$. This constant could then be used to predict the parameter value for which such nonlinear systems turn chaotic, that is, the parameter value ($P_\infty$) for which the period tends to infinity. He also recognized that each successive period doubling bifurcation is a replica of the bifurcation just before itself. This observation suggested a universal size scaling in the period doubling sequence, which led to the definition of the second Feigenbaum universal constant $\alpha$ [Feigenbaum, M. J., 1983]. The size of the chaotic attractor at a given parameter value is defined as the difference between the maximum and minimum values of the variable attained by the system for that parameter value. The demonstration of how the Fibonacci sequence appears within the Feigenbaum scaling of the period doubling cascade to chaos hinted at the possibility of a correlation between the size of the attractor (chaotic) at $P_\infty$ to the period 2 attractor [Linage et al., 2006]. Our work here lays down an argument for the existence of a universality relating the size of the chaotic attractor at $P_\infty$ to that of period 2. This discovery has immense potential in its application for designing experiments in the chaotic regime for systems exhibiting period doubling with the variation in system parameter values that is characteristic of an entire class of nonlinear phenomena.

2. Theory and calculations

We observed in plots of nonlinear systems following the period doubling route to chaos that the maximum widths of the bifurcations placed at the two extremities of the plot are related to the bifurcations preceding them by a factor of $\frac{1}{\alpha^2}$, at each period doubling occurring after period 4. If each of these outermost bifurcations (marked by bold strokes in the Figure 1) contributes a fixed ratio, assumed here to be $\gamma$, of
their maximum widths to the size of the chaotic attractor, we can construct an analytical expression for the size of the attractor at chaos ($P_\infty$).

Referring to Figure 1, let $d$ be the maximum size of the attractor when the period is 2. The maximum widths of the outermost period 4 branches are $d/\alpha$ and $d/\alpha^2$ [Feigenbaum, M. J., 1983]. The increment to the size of the attractor due to these branches would then be $\gamma d/\alpha$ and $\gamma d/\alpha^2$ respectively, as per our definition of $\gamma$ in the preceding paragraph. Using the Feigenbaum $\alpha$ to write all the subsequent widths of the outermost bifurcations in terms of $d$, the expression for the width of the attractor $w$ at the onset of chaos ($P_\infty$) can be written as a sum of the increments from these outermost bifurcations using our definition of $\gamma$. The resulting sum can be grouped into three terms as follows.

$$w = d + \gamma \frac{d}{\alpha}(1 + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \ldots) + \gamma d(\frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \frac{1}{\alpha^4} + \ldots).$$

(1)

Here, the second term comes from the increments from the outermost bifurcations linked to the lower branch of the period 2 orbit. The third term arises in the same way from the outermost bifurcations on the upper branch of the period 2 orbit. Simplification of the expression yields:

$$w = d(1 + \frac{\gamma}{(\alpha - 1)}).$$

(2)

We expect $\gamma$ to be a universal constant for all nonlinear systems that follow the period doubling route to chaos and for which the Feigenbaum universalities are valid. To verify this hypothesis, we calculated the ratio $w/d$ for a variety of maps and a couple of systems of ordinary differential equations that are nonlinear systems, and are known to exhibit the period doubling route to chaos.

We first found out the parameter value $P_1$ for which the system undergoes its first bifurcation and transitions from a period 1 value to period 2. Next, we looked for the parameter value $P_2$ at which the transition from period 2 to period 4 occurs. Subsequently, we found the difference $d$ between the maximum and minimum values of the variable just before the system bifurcates from period 2 to period 4. The value
of the parameter for period tending to infinity, $P_\infty$, was calculated using the Feigenbaum universal constant $\delta$ as described in standard textbooks on nonlinear dynamics.

$$P_\infty = (P_2 - P_1)(\frac{1}{\delta - 1}) + P_2. \quad (3)$$

The system was allowed to iterate many times at this parameter value, and the asymptotic difference $w$ between the maximum and the minimum values attained by the system variable during the course of these iterations was calculated. Finally, the ratio $\frac{w}{d}$ was evaluated.

3. Results

Table 1. Values of the ratio $\frac{w}{d}$ for different chaotic systems

<table>
<thead>
<tr>
<th>Nonlinear system</th>
<th>Ratio $\frac{w}{d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{n+1} = cx_n(1 - x_n)$</td>
<td>1.34569</td>
</tr>
<tr>
<td>$x_{n+1} = 0.5 - cx_n + x_n^3$</td>
<td>1.33045</td>
</tr>
<tr>
<td>$x_{n+1} = -cx_n\left(\frac{1}{2}x_n^3 - \frac{1}{2}x_n^2 + \frac{11}{22}x_n - \frac{3}{22}\right)$</td>
<td>1.43132</td>
</tr>
<tr>
<td>$x_{n+1} = c\sin(x_n)$</td>
<td>1.3497</td>
</tr>
<tr>
<td>$x_{n+1} = cx_n(1 - x_n^2)$</td>
<td>1.37096</td>
</tr>
<tr>
<td>$x_{n+1} = cx_n^3(1 - x_n)$</td>
<td>1.38322</td>
</tr>
<tr>
<td>$x_{n+1} = \sin(\pi x_n) + c$</td>
<td>1.35525</td>
</tr>
</tbody>
</table>

Maps:

Rossler Oscillator:
- $\dot{x} = -y - z$
- $\dot{y} = x + ay$
- $\dot{z} = bx - cz + xz$

Chemical Oscillator Model:
- $\dot{Y} = p(1 - \theta_{OH} - \theta_O) - qY$
- $\dot{\theta}_O = r\theta_{OH} - s\theta_O(1 - \theta_{OH} - \theta_O)$
- $\dot{\theta}_{OH} = Y(1 - \theta_{OH} - \theta_O) - [\exp(-\beta\theta_{OH}) + r]\theta_{OH} + 2s\theta_O(1 - \theta_{OH} - \theta_O)$

The results for various nonlinear maps and nonlinear systems of ordinary differential equations [Rossler, 1976] [Talbot & Oriani, 1985] [Montoya & Parmananda, 2009] have been compiled in the table. All the ordinary differential equation systems were solved using the Runge Kutta 4 method. The ratio calculated for all these systems returns an average value of 1.36788 with a standard deviation value 0.02971.

4. Conclusion

In conclusion, the numerical data obtained provides strong support to the hypothesis regarding the existence of a universality in the form of the ratio $\gamma$ as defined earlier. A direct consequence of this universality is the ability to predict the size of the attractor of a system in the chaotic regime with only the knowledge of the maximum size of the attractor when the system oscillates with period 2. This would have applications in designing experiments that seek to tap into the nonlinear behavior of the system in the chaotic regime armed with only limited information about its periodic behavior.

References

REFERENCES


