THE CRITICAL ISOThERM OF THE MODIFIED F MODEL*

Xin SUN**

Department of Physics, Fudan University, Shanghai, People’s Republic of China, and Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60201, USA

and

F.Y. WU

Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA

Abstract

On the basis of the linked-graph expansion theorem established by one of us, it is shown that the modified F model in a small electric field is related to a two-dimensional nearest-neighbor Ising model in an external magnetic field. Applying this relation to the critical isotherm we get $\delta = 7$, and confirm the scaling relation.

Résumé

Sur la base du théorème de développement en graphes liés établi par l’un d’entre nous, on montre que le modèle F modifié dans un champ électrique faible est relié à un modèle Ising à interaction entre plus proches voisins en deux dimensions soumis à un champ magnétique externe. En appliquant cette relation à l’isotherme critique, nous obtenons $\delta = 7$, et la relation d’échelle se trouve confirmée.

1. Introduction

One of the most fruitful lattice models in statistical physics is the vertex model of a ferroelectric\(^1\). The eight-vertex model is defined by placing arrows on the bonds of a square lattice such that there are even numbers of in and out arrows at each vertex. These eight configurations are shown in fig. 1.

Baxter\(^2\) has solved the symmetric eight-vertex model ($e_1 = e_2 = e_3 = e_4 = e_5 = e_6 = e_7 = e_8 = e_4$), which corresponds to a ferroelectric in zero external electric field. While the zero electric-field exponents of the eight-vertex model are found to vary continuously with the energy parameters $e_i$\(^3\), it is expected that other exponents will also vary with $e_i$ and that the scaling

---

* Supported in part by Fudan University (Shanghai), People’s Republic of China, and in part by the National Science Foundation through grants Nos. DMR 78-18808 and DMR 78-25708.
** Present address: Department of Physics, University of California, San Diego. La Jolla, California 92037.
relations will remain valid. In particular, we expect the relations
\[ \delta = \frac{2 - \alpha + \gamma}{2 - \alpha - \gamma}, \quad \alpha' + \beta(1 + \delta) = 2 \] to hold. Unfortunately, exact evaluation of the field exponent \( \delta \) is difficult for the eight-vertex model and consequently, there has been no convincing test of the scaling relations (1).

In this work we consider a special case of the eight-vertex model and show that scaling is indeed valid in this case. This is the modified F (MF) model proposed by Wu\(^6\). In the MF model the vertex energies are restricted by the relation
\[ \epsilon_1 + \epsilon_2 = \epsilon_3 + \epsilon_4. \] In zero electric field this model is equivalent to a nearest-neighbor Ising model, and from this equivalence Brascamp et al.\(^6\) have determined the following exponents for the MF model,
\[ \alpha = \alpha' = 0, \quad \beta = \frac{1}{4}, \quad \gamma = \gamma' = \frac{1}{3}, \quad \eta = \frac{1}{3}, \quad \nu = \nu' = 1. \] While these exponents indeed satisfy the scaling predictions, there has been no confirmation of the scaling relations (1). This is so because the procedure of Brascamp et al. does not lend itself to the determination of the exponent \( \delta \).

In the following we show that the MF model in a small external electric field is equivalent to a nearest-neighbor Ising model in an external magnetic field. This equivalence enables us to use the established results for the Ising model to deduce conclusions on the MF model. In particular, the value of \( \delta \) in the presence of an electric field can be obtained. The proof of the equivalence of the two models is based on a linked-graph theorem established by one of us\(^7\). In a preliminary report of this work\(^8\) the proof of the linked-graph
Theorem was not given. We give here the proof and show that the theorem leads to the desired equivalence.

2. Linked-graph expansion theorem

The linked-graph theorem upon which our discussion is based was originally developed for quantum spin systems and given in Chinese. Since most of the physicists here do not read Chinese, we now give a derivation of the theorem specializing to the Ising model.

Consider a nearest-neighbor Ising model whose Hamiltonian is

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}', \]  
\[ \mathcal{H}_0 = \sum_i c_i \sigma_i, \]  
\[ \mathcal{H}' = \sum_{ij} \mathcal{H}_{ij}', \quad \mathcal{H}_{ij}' = J \sigma_i \sigma_j, \]

where \( c_i \) are constants, \( \sigma_i (= \pm 1) \) is the spin located at the lattice point \( i \).

We use \( \langle A \rangle_{\mathcal{H}} \) to denote the ensemble average of quantity \( A \) over the Hamiltonian \( \mathcal{H} \), i.e.,

\[ \langle A \rangle_{\mathcal{H}} = \lim_{N \to \infty} \frac{\text{Tr}(e^{-\beta \mathcal{H}} A)}{\text{Tr} e^{-\beta \mathcal{H}}}. \]

**Theorem 1:** Let \( P_{lk} \) be a product of spins located at sites \( l \) and \( k \). Then the ensemble average \( \langle P_{lk} \rangle_{\mathcal{H}} \) can be expressed as an average over \( \mathcal{H}_0 \)

\[ \langle P_{lk} \rangle_{\mathcal{H}} = \langle e^{-\beta \mathcal{H}_0} P_{lk} \rangle_{\mathcal{H}_0}, \]

where the superscript \( c \) denotes a "linked-graph" ensemble average to be defined in the following.

**Proof:** First we have

\[ \langle P_{lk} \rangle_{\mathcal{H}} = \lim_{N \to \infty} \frac{\text{Tr}(e^{-\beta \mathcal{H}_0} P_{lk})}{\text{Tr} e^{-\beta \mathcal{H}_0}} \]
\[ = \lim_{N \to \infty} \frac{\langle e^{-\beta \mathcal{H}_0} P_{lk} \rangle_{\mathcal{H}_0}}{\langle e^{-\beta \mathcal{H}_0} \rangle_{\mathcal{H}_0}} \]
\[ = \lim_{N \to \infty} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \sum_i \cdots \sum_i \langle H_{i_{1}}', \ldots, H_{i_{n}}' P_{lk} \rangle_{\mathcal{H}_0} / \langle e^{-\beta \mathcal{H}_0} \rangle_{\mathcal{H}_0}. \]  

Next, we represent \( \mathcal{H}_{ij} \) graphically with a bond \( \cdots \) joining the neighboring
sites \(i\) and \(j\). Then

\[
\langle \mathcal{H}'_{i_1} \cdots \mathcal{H}'_{i_{ml}} P_{lkm} \rangle_{\mathcal{H}_0} = \frac{m!}{n-m!} \langle \mathcal{H}'_{i_1} \cdots \mathcal{H}'_{i_{ml}} \rangle_{\mathcal{H}_0} \langle \mathcal{H}'_{i_1} \cdots \mathcal{H}'_{i_{ml}} \rangle_{\mathcal{H}_0}.
\]  

(10)

In the first factor on the right-hand side of (10), all bonds are connected to either sides \(l\) or \(k\). In the second factor all bonds are separate from \(l\) and \(k\). Examples are given in fig. 2. The superscript \(c\) refers to this "connected", or linked-graph ensemble average.

Substituting (10) into (9), we obtain

\[
\langle P_{lk} \rangle_{\mathcal{H}} = \lim_{N \to \infty} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \langle \mathcal{H}^m P_{lk} \rangle_{\mathcal{H}_0} \langle \mathcal{H}^{n-m} \rangle_{\mathcal{H}_0} / (e^{-\beta \mathcal{H}})_{\mathcal{H}_0}
\]

\[
= \lim_{N \to \infty} \sum_{m=0}^{\infty} \frac{(-\beta)^m}{m!} \langle \mathcal{H}^m P_{lk} \rangle_{\mathcal{H}_0} \sum_{n=m}^{\infty} \frac{(-\beta)^{n-m}}{(n-m)!} \langle \mathcal{H}^{n-m} \rangle_{\mathcal{H}_0} / (e^{-\beta \mathcal{H}})_{\mathcal{H}_0}
\]

\[
= \langle e^{-\beta \mathcal{H}} P_{lk} \rangle_{\mathcal{H}_0}.
\]

Q.E.D.

Theorem 2: The free energy \(F(\mathcal{H})\) of the Hamiltonian \(\mathcal{H}\) can be expressed in terms of a linked-graph ensemble average as

\[
F(\mathcal{H}) = F(\mathcal{H}_0) - \frac{1}{\beta} \langle (e^{-\beta \mathcal{H}} - 1) \rangle_{\mathcal{H}_0}.
\]  

(11)

Proof: Define

\[
\mathcal{H}(\lambda) = \mathcal{H}_0 + \lambda \mathcal{H}',
\]  

(12)

so then

\[
F(\mathcal{H}(\lambda)) = -\frac{1}{\beta} \lim_{N \to \infty} \ln \text{Tr} e^{-\beta(\mathcal{H}_0 + \lambda \mathcal{H}')},
\]  

(13)

and

\[
\frac{\partial}{\partial \lambda} F(\mathcal{H}(\lambda)) = \langle \mathcal{H}' \rangle_{\mathcal{H}(\lambda)}.
\]  

(14)

Fig. 2. Example of linked-graph. (a) Linked graph. (b) A graph which is not linked.
Now applying theorem 1 to the ensemble average \( \langle \mathcal{H}^{e} \rangle_{\mathcal{H}(\lambda)} \), we obtain

\[
\frac{\partial}{\partial \lambda} F(\mathcal{H}(\lambda)) = \langle e^{-\lambda \mathcal{H}^{e}} \mathcal{H}^{e} \rangle_{\mathcal{H}_{0}}.
\]

Integrating both sides over \( \lambda \) from 0 to 1, we get

\[
F(\mathcal{H}) - F(\mathcal{H}_{0}) = \int_{0}^{1} \langle e^{-\lambda \mathcal{H}^{e}} \mathcal{H}^{e} \rangle_{\mathcal{H}_{0}} d\lambda = -\frac{1}{\beta} \langle (e^{-\mathcal{H}^{e}} - 1) \rangle_{\mathcal{H}_{0}}.
\]

This is precisely (11). Q.E.D.

3. Equivalence of the MF model with an Ising model

Consider the MF model with an external electric field \( E \) directed at 45° in the first quadrant. The vertex energies now become

\[
e_{1} = e_{1} + E, \quad e_{2} = e_{1} - E,
\]

\[
e_{3} = e_{4} = e_{2}, \quad e_{5} = e_{6} = e_{3}, \quad e_{7} = e_{8} = e_{4}.
\]

subject to (2).

It is well-known that the eight-vertex model is equivalent to an Ising model. The equivalent Ising Hamiltonian for the MF model (15) reads

\[
H(E) = \sum_{\alpha} \frac{1}{2} J_{\alpha} \left( \sum_{(ij)} \sigma_{i} \sigma_{j} + \sum_{(lm)} \tau_{l} \tau_{m} \right) + \frac{1}{2} E \sum_{(ij)} \sigma_{i} \tau_{l} + \frac{1}{2} (e_{1} + e_{2} + e_{3} + e_{4}),
\]

where

\[
J_{1} = \frac{1}{2} (e_{1} - e_{2} + e_{3} - e_{4}), \quad J_{2} = \frac{1}{2} (e_{1} - e_{2} - e_{3} + e_{4}).
\]

Here, we have divided the square Ising lattice into two sublattices whose spins are denoted by \( \sigma_{i} \) and \( \tau_{l} \) respectively. As shown in fig. 3, eq. (16) represents an Ising model with first- and second-neighbor interactions.

The polarization \( P(E) \) of the MF model is now computed from the Ising free energy

\[
F(H(E)) = -\frac{1}{\beta} \lim_{N \to \infty} \ln \text{Tr} e^{-\beta H(E)},
\]

as

\[
P(E) = -\frac{1}{N} \frac{\partial}{\partial E} F(H(E)).
\]
To facilitate discussion, write
\[ \Delta \sigma_i = \sigma_i - \langle \sigma_i \rangle_t, \quad \Delta \tau_i = \tau_i - \langle \tau_i \rangle_t, \quad (19) \]
and
\[ H(E) = H_1 + EH', \quad (20) \]

where
\[ H_1 = \sum_{\alpha} \frac{1}{2} J_{\alpha} \sum_{\langle ij \rangle} \sigma_i \sigma_j + h' \sum_i \sigma_i + \sum_{\alpha} \frac{1}{2} J_{\alpha} \sum_{\langle im \rangle} \tau_i \tau_m + h \sum_i \tau_i - N E \langle \sigma_i \rangle_t \langle \tau_i \rangle_t, \quad (21) \]
\[ H' = \frac{1}{2} \sum_{(\ell m)} \Delta \sigma_i \Delta \tau_i, \quad (22) \]
\[ h = 2E \langle \sigma_i \rangle_t, \quad h' = 2E \langle \tau_i \rangle_t, \quad (23) \]
and \( \langle \ldots \rangle_t \) denotes the average over \( H_1 \).

Now, \( H_1 \) is the Hamiltonian of two decoupled nearest-neighbor Ising models in external magnetic fields \( h \) and \( h' \). Since the sublattices \( \sigma_i \) and \( \tau_i \) are decoupled, we have
\[ \langle H' \rangle_t = 0. \quad (24) \]

Further, write
\[ H_1 = K_0 + K', \quad (25) \]
with
\[ K_0 = [2(\sigma_1 + J_2)(\sigma_i) + h'] \sum_i \sigma_i + [2(\sigma_1 + J_2)(\tau_i) + h] \sum_i \tau_i \]
\[ - \frac{1}{2} N (J_1 + J_2) \langle \sigma_i \rangle^2 + \langle \tau_i \rangle^2 - N E \langle \sigma_i \rangle_t \langle \tau_i \rangle_t, \quad (26) \]
\[ K' = \sum_{\alpha} \frac{1}{2} J_{\alpha} \left( \sum_{\langle \ell m \rangle} \Delta \sigma_i \Delta \sigma_j + \sum_{\langle \ell m \rangle} \Delta \tau_i \Delta \tau_m \right). \quad (27) \]
We now apply theorem 2 to $H_1$ and $H(E)$, or
\[ H(E) = K_0 + K' + EH'. \tag{28} \]
This leads to
\[
F(H(E)) = F(K_0) - \frac{1}{\beta} \langle (e^{-\beta(K+E'H')} - 1) \rangle_{K_0}, \tag{29}
\]
\[
F(H_1) = F(K_0) - \frac{1}{\beta} \langle (e^{-\beta K'} - 1) \rangle_{K_0}. \tag{30}
\]
Thus we get
\[
F(H(E)) - F(H_1) = -\frac{1}{\beta} \langle (e^{-\beta K'}(e^{-\beta E'H'} - 1)) \rangle_{K_0} \tag{31}
\]
when $E \neq 0$, since $F(H(E))$ and $F(H_1)$ are analytical functions of $E$ at $T_c$, the MF model reduces to the Ising model as $E \to 0$. Therefore (31) is convergent at $T_c$.

On the other hand, by applying theorem 1 to $(H')_t$, we obtain
\[
(H')_t = \frac{1}{2} \sum_{(ij)} \langle \Delta \sigma_i \Delta \tau_j \rangle = \frac{1}{2} \sum_{(ij)} \langle e^{-\beta K'} \Delta \sigma_i \Delta \tau_j \rangle_{K_0} = \langle e^{-\beta K'}(H') \rangle_{K_0}, \tag{32}
\]
using (24). Therefore
\[
\langle e^{-\beta K'}(H') \rangle_{K_0} = 0. \tag{33}
\]
Substituting (33) into (31), we get
\[
F(H(E)) = F(H_1) - \frac{1}{\beta} \sum_{n=2}^{\infty} E^n \frac{(-\beta)^n}{n!} \langle e^{-\beta K'} H'^n \rangle_{K_0}, \tag{34}
\]
or
\[
F(H(E)) = F(H_1) + O(E^2). \tag{35}
\]
Eq. (35) shows that the MF model (15) for a small electric field is equivalent to a nearest-neighbor Ising model $H_1$ with an external magnetic field $h = 2E\langle \sigma \rangle_t$ to the order of $O(E^2)$. This is our key result.

4. The critical isotherm

We now specialize (35) to the critical temperature $T_c$ for the small magnetic field $h$ and magnetization $m(h)$. Let the critical isotherm of a nearest-neighbor
Ising model be

\[ m(h') = \langle \sigma_i \rangle = A_m h^{r(1/\delta_m)} , \]
\[ m(h) = \langle \tau_i \rangle = A_m h^{1/\delta_m} , \]

where we have used the subscript \( m \) to denote the magnetic quantities. Eq. (23) now leads to

\[ h = 2E A_m h^{r(1/\delta_m)} , \quad h' = 2E A_m h^{1/\delta_m} . \]  

Therefore

\[ h = h' = (2A_m E)^{\delta_m/(\delta_m-1)} . \]  

This expression relates the magnetic field of the Ising model to the electric field in the MF model.

We can now compute the polarization per vertex according to (18) and (35). Also, using (21) and (23), we find

\[ P(E) = \frac{\partial}{\partial E} (Em^2(h)) - \frac{\partial h}{\partial E} m + O(E) = -[m(h)]^2 + O(E) . \]  

Eq. (38) says that, to the order of \( E \), the polarization of the MF model is equal to the square of the magnetization of a nearest-neighbor Ising model. Eq. (38) becomes exact for \( E = h = 0 \),

\[ P(0) = -[m(0)]^2 . \]  

This expression, which relates the spontaneous polarization of the MF model to the spontaneous magnetization of the Ising model, was first pointed out by Baxter and Kelland\textsuperscript{[10]}. Finally, we write the critical isotherm of the MF model as

\[ P(E) = A_e E^{1/\delta} . \]  

Eqs. (38), (36) and (37) lead to the relations

\[ \delta = \frac{1}{2}(\delta_m - 1) , \]  
\[ A_e = 2^{1/\delta} A_m^{2+1/\delta} . \]  

Using the established value of \( \delta_m = 15 \) for the two-dimensional Ising model, we then obtain

\[ \delta = 7 . \]  

This result confirms the scaling predictions and the scaling relations (1). In addition, eq. (42) relates the amplitudes of the magnetic and the electric critical isotherms of the MF model.
Acknowledgments

X. Sun is grateful to Professor C.-W. Woo for his assistance and encouragement. He also would like to thank the Department of Physics and Astronomy of Northwestern University for the hospitality during his visit there.

References

7) X. Sun, Fudan Acta 4 (1975) 37.