THE INFINITE-STATE POTTs MODEL
AND SOLID PARTITIONS OF AN INTEGER

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It has been established that the infinite-state Potts model in $d$ dimensions generates restricted partitions of integers in $d - 1$ dimensions, the latter a well-known intractable problem in number theory for $d > 3$. Here we consider the $d = 4$ problem. We consider a Potts model on an $L \times M \times N \times P$ hypercubic lattice whose partition function $G_{L, M, N, P}(t)$ generates restricted solid partitions on an $L \times M \times N$ lattice with each part no greater than $P$. Closed-form expressions are obtained for $G_{322P}(t)$ and we evaluated its zeroes in the complex $t$ plane for different values of $P$. On the basis of our numerical results we conjecture that all zeroes of the enumeration generating function $G_{L, M, N, P}(t)$ lie on the unit circle $|t| = 1$ in the limit that any of the indices $L, M, N, P$ becomes infinite.

1. Introduction

It has been recently established\(^1\) that the $q$-state Potts model in the $q \to \infty$ limit is intimately related to the problem of partitions of integers in number theory. Specifically, it was shown\(^2,3\) that the $d$ dimensional Potts model\(^4,5\) in the infinite-state limit generates $(d - 1)$-dimensional restricted partitions of integers.\(^5\) Using this equivalence and the known solutions of the enumeration problem\(^6\) for $d = 2, 3$, the infinite-state Potts model is solved\(^1\) on certain finite lattices in $d = 2, 3$. But the solution for the partition enumeration problem is open for $d > 3$. Here, we investigate this open problem for $d = 4$ by making use of the Potts equivalence. Specifically, we study zeroes of the enumeration generating function of restricted solid partitions, and show that their distribution approaches a unit circle as the size of the partitioned parts increases.

The consideration of zeroes of the partition function plays an important role in the analysis of phase transitions in statistical mechanics.\(^6,9\) However, the precise location of the zero distribution are known only in a very few instances. This includes the Ising lattice gas whose partition function zeroes lie on a unit circle in the complex fugacity plane.\(^8\) For the enumeration problem in $d = 2, 3$, alluded to in the above, one finds that the zeroes of the generating function also lie on a unit circle.\(^7\) But for $d > 3$ the zeroes of the generating function computed for small lattices are found to scatter, and their distribution does not appear to follow a regular pattern. On the other hand, the related Potts model has been solved,\(^7\) and the solution is consistent with the assumption that, in the thermodynamic limit (of infinite lattices), all partition function zeroes lie on a unit circle. This suggests
that a fruitful approach to the enumeration problem is to look into the zeroes of the generating function. This is the topic of this investigation.

We evaluate the generating function of restricted solid partitions on finite lattices and study the location of its zeroes. Our main finding, which is suggested by the Potts counterpart, is that the zeroes approach a unit circle as the size of the partitioned parts increases. This leads us to conjecture that the zeroes of the enumeration generating function of restricted solid partitions lie on a circle, when the size of partitioned parts, or equivalently the lattice size, becomes infinite.

2. The Potts model and restricted partitions

Restricted solid partitions can be generated by considering a Potts model. Consider a Potts model on a four-dimensional hypercubic lattice of size \( L \times M \times N \times P \). The lattice sites are specified by coordinates \( i, j, k, p \), where \( 1 \leq i \leq L, 1 \leq j \leq M, 1 \leq k \leq N, \) and \( 1 \leq p \leq P \). Introduce an extra site which is connected by edges to every site in the hyperplanes \( i = L, j = M, k = N \) and \( p = P \). The resulting lattice contains \( LMNP + 1 \) vertices and \( 4LMNP \) edges.

The high-temperature expansion of the Potts partition function assumes the form:

\[
Z_{LMNP}(q, x) = \sum_{\text{bond config.}} x^b q^{n_b} / q^{E_b},
\]

where \( x = (e^\beta - 1) / q^{1/4} \), \( b \) and \( n_b \) are, respectively, the numbers of bonds and connected clusters, including isolated points. In the large \( q \) limit the leading terms in (1) are of the order of \( q^{LMNP+1} \). One introduces the reduced partition function

\[
G_{LMNP}(x) \equiv \lim_{q \to \infty} q^{-(LMNP+1)} Z_{LMNP}(q, x),
\]

which is a polynomial of degree \( LMNP \) in \( x^4 \).

It has been shown\(^3\) that the reduced partition function \( G_{LMNP}(t) \) is precisely the generating function of restricted solid partitions of a positive integer into a sum of parts on an \( L \times M \times N \) cubic lattice, with each part no greater than \( P \). The generating function for the solid partition is defined by

\[
G_{LMNP}(t) = 1 + \sum_{n=1}^{LMNP} A_n(L, M, N) t^n,
\]

where \( A_n(L, M, N) \) is the number of distinct ways that a positive integer \( n \) is partitioned into the sum of nonnegative integers \( m(i, j, k) \),

\[
n = \sum_{i=1}^{L} \sum_{j=1}^{M} \sum_{k=1}^{N} m(i, j, k),
\]

subject to

\[
0 \leq m(i, j, k) \leq P.
\]
and

\[ m(i, j, k) \leq \{ m(i - 1, j, k), m(i, j - 1, k), m(i, j, k - 1) \}. \tag{6} \]

We point out that, despite the apparent asymmetric footing, \( G_{LMNP}(t) \) is actually symmetric in the 4 indices \( L, M, N \) and \( P \), a fact which is obvious from the Potts equivalence.\(^1\)

The explicit expression of \( G_{LMNP}(t) \) for general \( \{ L, M, N, P \} \) is not known. However, for \( L = M = N = 2 \) MacMahon\(^6\) has obtained a closed-form expression given by

\[ G_{2222}(t) = \sum_{i=0}^{4} L_i \frac{(t)(P+3-i)}{(t)(P-1)}, \tag{7} \]

with

\[
\begin{align*}
L_0 &= 1 \\
L_1 &= 2t^2 + 2t^3 + 3t^4 + 2t^5 + 2t^6 \\
L_2 &= t^6 + 3t^8 + 4t^7 + 8t^8 + 4t^9 + 3t^{10} + t^{11} \\
L_3 &= 2t^{10} + 2t^{11} + 3t^{12} + 2t^{13} + 2t^{14} \\
L_4 &= t^{16},
\end{align*}
\]

and

\[ (t)_m = \prod_{p=1}^{m} (1 - t^p), \quad m \geq 1. \tag{8} \]

Before we proceed further, we first cast (7) into an alternate form which is more suggestive.

For \( d = 2 \) and 3 the partition generating functions for similarly defined line and planar partitions assume the form\(^6\)

\[ G_{LM}(t) = \frac{(t)(L+M)}{(t)(L)(M)} \tag{10} \]

\[ G_{LMN}(t) = \frac{(t)(L+M+N)(L)(M)(N)}{(t)(L)(M)(N)} \tag{11} \]

where

\[ (t)_m = \prod_{p=1}^{m-1} (1 - t^p), \quad m \geq 2. \tag{12} \]

The expression which straightforwardly generalizes (10) and (11) to \( d = 4 \) is

with

$$\{t\}_m = \prod_{p=2}^{m-1} \{t\}_p, \quad m \geq 3.$$  \hspace{1cm} (14)

Note that the $LMNP + 1$ zeroes of $G^{(0)}_{LMNP}(t)$ lie on the unit circle $|t| = 1$. This suggests one to write

$$G_{LMNP}(t) = G^{(0)}_{LMNP}(t) + C_{LMNP}(t),$$  \hspace{1cm} (15)

where $C_{LMNP}(t)$ is a "correction" to the straightforward extension (13). For $L = M = N = 2$ we find that (7) can indeed be be rewritten in the form of (15) with

$$C_{222P}(t) = \left( \frac{t^2(t+1)^2(t^2 - 2t^2 + t^2 - 2t + 1)}{t^2 + t + 1} \right) \left( \frac{\langle t \rangle_{P+2}}{\langle t \rangle_{P-2}} \right).$$  \hspace{1cm} (16)

![Diagram](image)

(a) \hspace{2cm} (b) \hspace{2cm} (c)

Fig. 1. Zeros of $G_{222P}$ for (a) $P = 2$, (b) $P = 6$, and (c) $P = 10$.

3. **Zeros of the generating function**

We now use (7), or equivalently (15), to evaluate the zeroes of $G_{222P}(t)$. Selected results for $P = 2, 6, 10$ are shown in Fig. 1. It is seen that, while for $P = 2$ none of the zeroes lie on the unit circle $|t| = 1$, more zeroes are found on the unit circle as $P$ increases. To measure quantitatively the deviation of the zeroes from the unit circle, we have computed $d_1$, the average distance, and $d_2$, the root-mean-square distance, of the zeroes from the unit circle given by

$$d_1 = \frac{1}{LMNP} \sum_{i=1}^{LMNP} (|t_i| - 1)$$

$$d_2 = \frac{1}{LMNP} \sqrt{\sum_{i=1}^{LMNP} (|t_i| - 1)^2}.$$  \hspace{1cm} (17)
Fig. 2. Distance to the unit circle $|t| = 1$ for $G_{222P}(t)$.

Fig. 3. Distance of zeroes to the unit circle $|t| = 1$ for $G_{322P}(t)$. 
Plots of $d_1$ and $d_2$ are shown in Fig. 2. It is clear that both $d_1$ and $d_2$ extrapolate to zero in the limit of $P \to \infty$.

We have also generated explicitly the generating function $G_{22P}(t)$ for $P = 3$ to 14, and evaluated their zeroes. The results are shown in Fig. 3. Again, it is seen that both $d_1$ and $d_2$ extrapolate to zero, namely, all zeroes lie on the unit circle, as $P \to \infty$. This leads us to conjecture that, quite generally and making use of the indices symmetry, zeroes of the enumeration generating function $G_{LMNP}(t)$ of restricted solid partitions lie on a unit circle in the limit that any of the indices $(L, M, N, P)$ becomes infinite.

4. Summary and conclusion

We have obtained closed-form expressions for the enumeration generation function $G_{22P}(t)$ for restricted solid partitions of an integer on a $2 \times 2 \times 2$ lattice into parts which are equal to or less than $P$. We have also evaluated the zeroes of the generating functions $G_{22P}(t)$ and $G_{33P}(t)$ for fixed values of $P$, and found that they approach the unit circle $|t| = 1$ as the value of $P$ increases. On the basis of our finding and the known result of a related Potts model, we conjecture that all zeroes of the enumeration generating function $G_{LMNP}(t)$ lie on a unit circle in the limit that any of the indices $(L, M, N, P)$ becomes infinite.

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References

4. For a review on the Potts model and its physical relevance, see F. Y. Wu, Rev. Mod. Phys. 54, 235 (1982).