STAR-STAR RELATION FOR THE POTTS MODEL

C. KING1,3 and F. Y. WU1,3

1Department of Mathematics, 2Department of Physics
3Center for Interdisciplinary Research in Complex Systems
Northeastern University, Boston, Massachusetts 02115, USA

A star-star transformation is introduced for the Potts model as a generalization of the Onsager star-triangle relation. By considering a special limit of the star-star relation, we deduce a dual transformation relation for the Potts model on the 3-12 lattice. We also obtain numerical bounds on the critical point of the Potts model on the kagomé lattice, a long unsolved problem.

1. Introduction

Transformation relations connecting difference lattices provide a powerful technique for analyzing lattice spin models.1 The most prominent example is the Onsager star-triangle relation2 for the Ising model, a transformation which has its origins in the Δ-Y relation of Kennelly3 for electrical circuits, where the reference to a star and a triangle was first made. The Onsager star-triangle relation contains essentially all the information needed to solve the two-dimensional Ising model, and in fact, as Baxter and Eting2 have shown, from the star-triangle relation alone one can deduce the Onsager solution and calculate spin correlation functions.

Most of the known transformation relations of spin models apply only to the Ising model (see, for example, the review by Syozi4). The star-triangle relation in its usual form can be extended to the Potts model5 only at a fixed temperature, so its usefulness is greatly reduced.

In this paper we introduce a star-star relation for the Potts model as a generalization of the usual star-triangle relation, and consider some of its implications. In particular we show that it leads to a “dual” transformation for the Potts model on the 3-12 lattice. We also use the star-star relation to deduce bounds on the critical temperature of the Potts model on the kagomé lattice, thus addressing a long unsolved problem.

2. The star-star relation

The Onsager star-triangle relation equates the partition functions of two small lattices with three terminal spins, one connected in the form of a “Y” (star) and one in the form of a “Δ” (triangle). For the isotropic Ising model, this requires two independent conditions to be satisfied. Since there are also two unknowns, a solution exists at all temperatures. For the q-state Potts model with q ≥ 3, there are three independent conditions to be satisfied. Hence the equations possess a solution only
at a special temperature (which turns out to be the critical point of the honeycomb and triangular lattices). This limitation can be overcome by generalizing the "Y" and "Δ" lattices to "stars" which include two independent interactions. This can be done in several ways. In this paper we will consider the two configurations shown in Fig. 1, where the shaded triangle in the second diagram represents a 3-site coupling. Other star configurations involving two independent interactions can also be considered, but we do not pursue the issue here.

![Fig. 1. A star-star transformation. Solid circles denote summed spin states.](image)

Let the three external spins be in states \( s = (σ_1, σ_2, σ_3) \), where \( σ = 1, \ldots, q \). As a shorthand we write \( δ_{ab} \) for \( δ_{σ_a, σ_b} \), and \( δ_{abc} \) for \( δ_{σ_a, σ_b, σ_c} \). The partition functions for the lattices shown in Fig. 1 are respectively

\[
Z_1(P, W; s) = \sum_{σ_1, σ_2, σ_3} e^{P(δ_{12} + δ_{23} + δ_{31}) + W(δ_{13} + δ_{23} + δ_{31})}
\]

(1)

\[
Z_2(N, M; s) = \sum_{σ_1, σ_2, σ_3} e^{N(δ_{12} + δ_{23} + δ_{31}) + M(δ_{13} + δ_{23} + δ_{31})}
\]

(2)

The star-star relation is the condition that, up to an overall factor, the two partition functions are equal for all terminal spins \( s \). By symmetry both of these partition functions assume the form

\[
Z_i = A_i + B_i(δ_{12} + δ_{23} + δ_{31}) + C_iδ_{123} \quad (i = 1, 2).
\]

(3)

From (3) it follows that the star-star relation leads to three independent conditions. Since each partition function has two independent couplings in addition to an overall factor, there are always solutions.

If we write

\[
p = e^P - 1, \quad w = e^W - 1, \quad m = e^M - 1, \quad n = e^N - 1,
\]

(4)
then it is straightforward to verify that we have (for $q \geq 3$)
\[ A_1(p, w) = p^3 + 3(q + 2w)p^2 + (q^2 + 3qw + 3w^2 + w^3)(3p + q) \]
\[ B_1(p, w) = wp^2 + (qw + 3w^2 + w^3)p^2 \]
\[ C_1(p, w) = (3w^2 + w^3)p^2 \]
\[ A_2(m, n) = (n + q)^3 + m(3n + q) \]
\[ B_2(m, n) = mn^2 \]
\[ C_2(m, n) = mn^3. \] (5)

In this paper we will explore the consequences of the star-star relation in the limit of $P \to \infty$, or $p \to \infty$. The equality of the two partition functions leads to the following three equations:
\[ F \left[ (n + q)^3 + m(3n + q) \right] = 1 \]
\[ Fmn^2 = w \]
\[ Fmn^3 = 3w^2 + w^3 \] (6)

where $F$ is an overall factor. Solving (6) for $m, n, F$ and defining
\[ h(w) = \frac{(q + w(w + 3))^3}{w^2(w + 3) - q} \] (7)

one finds
\[ m = h(w) \]
\[ n = w(w + 3) \]
\[ F = [w(w + 3)^2h(w)]^{-1}. \] (8)

Eq. (8) relates $m$ and $n$ through the parameter $w$, indicating that they are not independent (a consequence of $P \to \infty$).

3. Dual relation for the 3-12 lattice

The transformation (8) implies a "dual" relation for the 3-12 lattice. The 3-12 lattice is illustrated in Fig. 2(a). The Potts model on this lattice is defined by two independent couplings ($J, W$) on the two types of edges shown. Its critical manifold has been studied, but the exact critical point is known only in the limiting case $W \to \infty$, where it reduces to the honeycomb lattice.

By applying the star-star relation (8) to each triangle, the 3-12 model with couplings ($J, W$) is mapped to a decorated honeycomb lattice, where each vertex of the underlying honeycomb lattice is replaced by three spins interacting with a three-site coupling $M$, and each edge is decorated by two extra spins with couplings $N, J, N$ in series. The three couplings $N, J, N$ can be further replaced by a single two-site coupling $L$ given by
\[ T(L) = T(N)T(J)T(N), \] (9)
where

\[ T(L) \equiv 1 + q/(e^L - 1). \] (10)

The 3-12 Potts model is thus transformed into a honeycomb model with two-site couplings \( L \) and three-site couplings \( M \) as shown in Fig. 2(b). We shall refer to the latter as the 2-3 honeycomb lattice. The couplings \( L \) and \( M \) are given in terms of \( J \) and \( W \) using (9) and (8).

Our first observation is that the inverse transformation is not unique. Starting from a 2-3 honeycomb lattice with fixed \( L \) and \( M \), it is readily verified that in the region

\[
\frac{w^3(w+3)}{m} > q, \\
m > \frac{(2\sqrt{q} + 3)^3}{(1 + 2/\sqrt{q})}
\] (11)

de the equation \( h(w) = m \) has two real solutions \( w_1 \) and \( w_2 \). It is also seen that there exist couplings \( J_1 \) and \( J_2 \) such that

\[ T(N_1)T(J_1)T(N_1) = T(N_2)T(J_2)T(N_2), \] (12)

where \( e^{\theta_i} = 1 + w_i(w_i + 3), i = 1, 2 \). Using (9), we see that the 3-12 lattices with couplings \( \{J_1, W_1\} \) and \( \{J_2, W_2\} \), where \( w_i = e^{\theta_i} - 1, i = 1, 2 \), are transformed to the same 2-3 honeycomb lattice. This defines a dual transformation relation for the 3-12 lattice, implying that the free energies of the two models with \( \{J_1, W_1\} \) and \( \{J_2, W_2\} \)

---

Fig. 2. A unit cell of (a) the 3-12 lattice, (b) the 2-3 honeycomb lattice.
are simply related. Particularly, the critical manifold of the 3-12 model must be invariant under this mapping. We note that the fixed set of the transformation is \( w = e^W - 1 = \sqrt{q} \) for all \( J \).

4. Bounds for the critical coupling of the kagomé model

The star-star relation (8) can be used to obtain bounds on the critical coupling of the kagomé model, a long unsolved problem. The kagomé model with coupling \( W \) is a special case of the 3-12 model in Fig. 2(a), obtained by setting \( J = \infty \). Applying the star-star relation (8) as depicted in Fig. 1 to each triangle of the kagomé lattice, one obtains again a 2-3 honeycomb lattice. The resulting 2-3 honeycomb lattice has couplings \( L > 0 \) and \( M > 0 \) (ferromagnetic) with \( M \) given by (8) and \( L \) by

\[
T(L) = [T(N)]^2
\]

which is (9) with \( T(J) = 1 \) (\( J = \infty \)).

Now Potts models with ferromagnetic couplings satisfy the Fortuin-Kasteleyn-Griffiths (FKG) inequalities,\(^9,10\) which implies in particular that the magnetization is an increasing function of the couplings (this can be seen by noting that the magnetization is the percolation probability in the random cluster representation). For the 2-3 honeycomb model the critical value \( L_c \) is known exactly when \( M = \infty \), when it becomes the usual honeycomb model. It can be seen from (11) and (13) that \( M, N, L \) are all ferromagnetic for \( W > 0 \). Therefore the critical value of \( L \) cannot be smaller than \( L_c \) for finite \( M \). Applying this reasoning to the values of \( (L, M) \) in (8) and (9), we deduce the following lower bound for the critical coupling \( w_c \) of the kagomé model:

\[
w_c \geq w_{18}
\]

where \( w_{18} \) is the solution of

\[
\left[ 1 + \frac{q}{w(w + 3)} \right]^2 = 1 + \frac{q}{e^{L_c} - 1}
\]

and \( L_c \) is the critical coupling of the honeycomb model given by\(^11\)

\[
1 + \frac{q}{e^{L_c} - 1} = 2 \cos \left[ \frac{2}{3} \cos^{-1} \left( \frac{\sqrt{q}}{2} \right) \right], \quad q = 3, 4
\]

\[
= 2 \cosh \left\{ \frac{2}{3} \ln \left[ \frac{\sqrt{q}}{2} + \left( \frac{q}{4} - 1 \right)^{1/2} \right] \right\}, \quad q \geq 4.
\]

Values of \( w_{18} \) computed using (15) and (16) are shown in Table 1 for \( q = 2, 3, 4, 5 \).

In fact we can improve this bound by noting that (i) the FKG inequality implies that the magnetization increases if we send the coupling \( M \to \infty \) on any subset of lattice sites (while leaving the couplings of the other subset of lattice sites fixed), and (ii) the critical coupling is known for the 2-3 honeycomb model where \( M \) is
set to infinity on alternating vertices of the underlying honeycomb lattice.\textsuperscript{12,13} This yields the following improved lower bound

\[ w_c \geq w_{2q} > w_{1q}, \]

where \( w_{2q} \) is the solution of the known critical condition\textsuperscript{8,13}

\[ qA_1(n, w) = C_1(n, w), \]

or, explicitly,

\[ q \left[ n^2 + 3(q + 2w)n^2 + (q^2 + 3qw + w^2)(3n + q) \right] = wn^4. \]

Numerical results of \( w_{2q} \) for \( q = 2, 3, 4, 5 \) are shown in Table 1. It is seen that the resulting bounds are surprisingly good when compared with numerical estimates of \( w_c \), obtained by Monte Carlo renormalization group analysis,\textsuperscript{14} which are also given in Table 1. (The values for \( q = 2 \) are given for comparison purposes. The exact value of \( w_c \) for \( q = 2 \) is known\textsuperscript{2} to be \( \sqrt{3} + 2\sqrt{3} - 1 = 1.54246... \) Improved numerical estimates are underway and results will be reported elsewhere.

<table>
<thead>
<tr>
<th>( q )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_{1q} )</td>
<td>1.4288</td>
<td>1.7153</td>
<td>1.9066</td>
<td>2.1605</td>
</tr>
<tr>
<td>( w_{2q} )</td>
<td>1.4840</td>
<td>1.7931</td>
<td>2.0494</td>
<td>2.3724</td>
</tr>
<tr>
<td>( w_{2qc} )</td>
<td>1.5868</td>
<td>1.9293</td>
<td>2.1618</td>
<td>2.3942</td>
</tr>
</tbody>
</table>

Acknowledgements

The work by FYW has been supported in part by NSF Grant DMR-9313648.

References

11. See, for example, F. Y. Wu, Rev. Mod. Phys. 54, 235 (1982).