Transformation Properties of a Spin-1 System in Statistical Physics

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In this paper we discuss transformation properties of the spin-1 Blume-Emery-Griffiths (BEG) model arising in statistical physics. It is shown that the BEG model can always be transformed into either a spin-1/2 Ising model or a 3-state Potts model. It is also shown that there exist other transformations relating the BEG model with spin-1/2 king models, valid in special subspaces of the parameter space. Physical properties of the BEG model that can be drawn from the equivalence are discussed. We also present a formulation of the BEG model relating it to a 3-state vertex model.

I. INTRODUCTION

In the consideration of the thermodynamics of multi-component systems, it is often convenient to formulate the system as a spin model and apply the method of statistical mechanics. Of particular interest is a spin-1 model, the Blume-Emery-Griffiths (BEG) model, used widely in describing a variety of physical phenomena, ranging from magnetic transitions in certain magnetic compounds, the A-transition and phase separation in He³-He⁴ mixtures, to the phase changes in a microemulsion. However, most of the studies in the past have been carried out using approximate schemes, including the renormalization group, mean-field analyses, and Monte Carlo simulations. In contrast, there exist a large body of transformation properties of the BEG model which are exact and useful in obtaining rigorous results. In this paper we present these transformation properties as well as physical implications that can be deduced from these relations.

The BEG model is defined by the (reduced) Hamiltonian

$$-\beta H = J \sum_{\langle ij \rangle} S_i S_j + K \sum_{\langle ij \rangle} S_i^2 S_j^2 - D \sum_i S_i^2 + H \sum_i S_i$$

where \(\beta = 1/kT\), \(S_i = 0, \pm 1\), \(J\) and \(K\) the respective polar and quadrupolar interactions, and \(A\) and \(H\) the respective crystal and direct fields. As revealed by mean-field analyses, the critical behavior of the BEG model is extremely complicated involving the occurrence of a variety of multi-critical phenomena accompanied with the onset of first- and second-order transitions. As we shall see, some of its critical behaviors can be deduced from a consideration of the transformation properties.
II. EQUIVALENCES WITH SPIN-1/2 SYSTEMS

The spin-1/2 Ising model has been in the forefront of research for many decades. It is therefore extremely useful, whenever possible, to cast lattice models in the form of a spin-1/2 system, and to see what can be said of the physical properties as a consequence.

**Equivalence 1**

Any spin-1 Hamiltonian with spin variables \( \mathbf{S} = 0, \pm 1 \) can be written as a spin-1/2 model by writing \( \mathbf{S} = (a + \tau)/2 \), where \( a, \tau = \pm 1 \). The double-counting of the \( \mathbf{S} = 0 \) state can be compensated by introducing in the partition summation the identity \(^{12}\)

\[
\sum_{S=0,\pm 1} f(S) = \sum_{\sigma=\pm 1} 2^{(-1+\sigma)/2} f\left( \frac{1}{2}(\sigma + \tau) \right). \tag{2}
\]

In this way, one can think of a spin-1 system as comprising of two identical spin-1/2 systems superimposed upon each other and coupled by an additional interaction in the form of \( \sum_i [(-1 + a\tau_i)\ln 2]/2 \). Using (2), we find, after combining all relevant terms, the following effective spin-1/2 Hamiltonian:

\[
-\beta H_{\text{eff}} = \sum_{\langle ij \rangle} E_{ij} + \frac{1}{2} H \sum_i (\sigma_i + \tau_i), \tag{3}
\]

where \( E_{ij} \) is a nearest-neighbor (reduced) energy given by

\[
E_{ij} = \frac{1}{4} \left[ J(\sigma_i + \tau_i)(\sigma_j + \tau_j) + K(\sigma_i \tau_i)(\sigma_j \tau_j) \right] + J_1(\sigma_i \tau_i + \sigma_j \tau_j) \tag{4}
\]

\( q \) being the coordination number of the lattice.

This equivalence permits us to draw useful information on the existence and nature of the transitions in the BEG model. For the ferromagnetic \( (J > 0) \) spin-1 Ising model, for example, we have \( K = A = 0, J_1 = (\ln 2)/2q > 0 \), thus the Lee-Yang circle theorem \(^{13}\) holds, namely, zeros of the partition function lie on the unit circle in the complex \( z = e^{-\beta H/\mathcal{K}} \) plane. In the general case we apply a generalized Lee-Yang theorem due to Suzuki and Fisher \(^{14}\) and consider the 4 spins \( \{\sigma_i, \tau_i, \sigma_j, \tau_j\} \) of neighboring sites \( \{i, j\} \) and the associated energy \( E_{ij} = E_{\sigma_i \sigma_j, \tau_i \tau_j} \) as building blocks. We have
Theorem (B) of Suzuki and Fisher, given in the form of (2.54) of Ref. [14], now says that zeros of the BEG partition function in the complex \( z = e^{-H/kT} \) plane lie on the unit circle in the regime

\[
\sinh J > e^{-2J_1}(e^{-2J_1} + 2e^{-K/2}),
\]

(6)

\( J_1 > 0. \)

This is a very strong statement saying that, in the regime (6) which is realized at sufficiently low temperatures in appropriate regimes of the parameter space, a phase transition can occur only at \( H = 0 \), thus ruling out the occurrence for \( H \neq 0 \).

**Equivalence \( J = 0 \)**

For \( J = 0 \), there exists another equivalence with a spin-1/2 Ising model. When \( J = 0 \), we write \( \sigma = 2S^2 - 1 \) and use, in place of (2), the identity

\[
\sum_{\sigma = \pm 1} e^{HS} f(S^2) = \sum_{\sigma = \pm 1} (2 \cosh H)^{(\sigma+1)/2} \left( \frac{\sigma + 1}{\sigma} \right)^{1/2}
\]

(7)

Then, besides an overall constant which does not concern us, the partition function of the BEG model becomes that of a spin-1/2 Ising model with interactions \( K/4 \) and an external magnetic field

\[
L = \frac{1}{2} \ln(2 \cosh H) - 4 \frac{K}{4} + \frac{1}{4} q K.
\]

(8)

This equivalence, first pointed out by Griffiths[15] for \( \Lambda = H = 0 \), can also be deduced from (4) directly by renaming \( \sigma_i \) as \( \sigma_l \) and identifying \( J_1 \) as \( L/q \). For \( K > 0 \), this leads to the existence of a first-order surface in \( L = 0, K > K_\epsilon \) bounded by a critical line \( K = K_\epsilon \), where \( K_\epsilon \) is the
critical point of the equivalent spin-1/2 Ising model.\textsuperscript{16,17} Implications in the case of $K < 0$ have also been discussed.\textsuperscript{16}

**Equivalence 3** ($K = -\ln \cosh J, q = 3$)

In the case of coordination number $q = 3$ and in the subspace

$$A I = -\ln \cosh J,$$

the BEG model (1) is reducible to a spin-1/2 Ising model through yet another consideration.\textsuperscript{18-21}

This equivalence is deduced by writing, for $K = -\ln \cosh J$,

$$e^{J S_i S_j + K S_i^z S_j^z} = 1 + z S_i^z S_j^z,$$

where

$$z = \tanh J.$$  \hspace{1cm} (11)

We expand the product $\prod_{ij} e^{J S_i S_j}$ in the partition summation after substituting (10) for the nearest-neighbor Boltzmann factors, and for each term in the expansion for which the factor $1$ or $z S_i S_j$ is taken, we draw respectively a dotted or solid line over the corresponding lattice edge. Then, the BEG partition function generates graphs on the underlying lattice. The associated weights to lattice sites (vertices) are obtained by carrying out the summations $\sum_{S_i=0, \pm 1}$. This procedure leads to a vertex model on the underlying lattice. For lattices of coordination number $q = 3$, this yields an eight-vertex model with vertex configurations shown in Fig. 1 and weights

\begin{align*}
a &= \sum_{S_i=0, \pm 1} e^{-\Delta S_i^z + H S_i} = 1 + 2e^{-\Delta \cosh H}, \\
b &= z^{1/2} \sum_{S_i=0, \pm 1} S_i e^{-\Delta S_i^z + H S_i} = 2z^{1/2} e^{-\Delta \sinh H}, \\
c &= z \sum_{S_i=0, \pm 1} S_i^2 e^{-\Delta S_i^z + H S_i} = 2z e^{-\Delta \cosh H}, \\
d &= z^{3/2} \sum_{S_i=0, \pm 1} S_i^3 e^{-\Delta S_i^z + H S_i} = 2z^{3/2} e^{-\Delta \sinh H}.
\end{align*}  \hspace{1cm} (12)

Since the 8-vertex model is completely equivalent to an Ising model,\textsuperscript{21,22} it follows that the BEG model is equivalent to a spin-1/2 Ising model. This property has proven to be useful in obtaining

![FIG. 1. Vertex configurations of the S-vertex model and the associated vertex weights.](image)
the phase diagram of the BEG model.\textsuperscript{21}

III. REDUCTION TO A 3-STATE VERTEX MODEL

Another formulation of the BEG model is its equivalence with a 3-state vertex model.\textsuperscript{23} To see this equivalence, we start from the generally valid identity

\[ e^{J S_i S_j + K S_i^2 S_j^2} = 1 + z S_i S_j + t S_i^2 S_j^2 \]  \hspace{1cm} (13)

where

\[ t = e^K \cosh J - 1, \quad z = e^K \sinh J. \] \hspace{1cm} (14)

Again, use (13) for the neighboring Boltzmann factors and expand the product $\prod_{<ij>}$ in the partition summation. For each term in the expansion for which the factor $1, z S_i S_j, t S_i^2 S_j^2$ is taken, we draw respectively a dotted, heavy, or thin line over the corresponding lattice edge. Then, this procedure generates graphs on the underlying lattice for which each lattice edge can be in one of three states. This yields a 3-state vertex model.

In the case of coordination number $q = 3$, for example, this 3-state vertex model is a 27-vertex model whose vertex configurations are shown in Fig. 2. Here, $\omega_{ijk}$ is the weight of a

\begin{align*}
\omega_{300} & \quad \omega_{030} & \quad \omega_{003} & \quad \omega_{111} & \quad \omega_{210} \\
\omega_{012} & \quad \omega_{102} & \quad \omega_{120} & \quad \omega_{021} & \quad \omega_{201}
\end{align*}

FIG. 2. Vertex configurations of the 27-vertex model and the associated vertex weights. Symmetric vertex configurations having equal weights are not shown.
vertex with respective $i,j,k$ dotted, heavy, and thin lines. Using the identities in (12), we find

$$\omega_{003} = 1 + e^{\Delta/2} \cosh H, \quad \omega_{201} = t^{1/2}, \quad \omega_{102} = t,$$

$$\omega_{023} = z^{3/2}, \quad \omega_{120} = z, \quad \omega_{012} = z^{1/2},$$

$$\omega_{030} = z^{3/2}, \quad \omega_{210} = t^{1/2}, \quad \omega_{102} = z^{1/2}.$$

Here, $h = \tanh H$ and we have multiplied a factor $e^{\Delta/2} \cosh H$ throughout. The equivalence (15) has previously been written down for $H = h = 0$, and utilized to obtain a closed-form expression of the critical surface of the BEG model.

**IV. EQUIVALENCE WITH A POTT S MODEL**

The BEG model (1) is also related to a 3-state Potts Hamiltonian

$$-\beta H_p = K \sum_{\langle ij \rangle} \delta_{K} (S_i, S_j) + M_p \sum_{\langle ij \rangle} \delta_{K} (S_i, 0) \delta_{K} (S_j, 0)$$

$$+ L_p \sum_{i} \delta_{K} (S_i, 0),$$

where $S_i = 0, \pm 1$, and $M_p, L_p$ are external fields. Writing

$$\delta_{K} (S_i, S_j) = 1 + \frac{1}{2} S_i S_j + \frac{3}{2} S_i^2 S_j^2 - S_i^2 - S_j^2,$$

$$\delta_{K} (S_i, 0) = 1 - S_i^2,$$

we can relate (16) and (1) with

$$\beta = 2 J,$$

$$M_p = K - 3 J,$$

$$L_p = \Lambda + q (J - K).$$

Particularly, for the standard Potts model $M_p = L_p = 0$, we obtain

$$K = 3 J, \quad \Lambda = 2 q J.$$

This implies that the special case (19) of the BEG model is equivalent to a standard 3-state Potts model, hence possessing the critical exponents $\alpha = 1/3, \nu = 5/6$ in two dimensions.
REFERENCES

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