RIGOROUS RESULTS ON THE ANISOTROPIC 
TRIANGULAR ISING MODEL

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We consider the Ising model on the triangular lattice with anisotropic pair interactions, staggered triplet interactions, and an external magnetic field. Using the Yang-Lee zero consideration and a star-triangle transformation which eliminates the triplet interactions, we obtain regimes in the parameter space which are free of phase transitions.

1. Introduction

The Ising model on the triangular lattice with two- and three-spin interactions has attracted considerable attention in the past years. The lattice is characterized by the (reduced) Hamiltonian

$$-\beta H = \sum_{nm} K_{nm} \sigma_i \sigma_j + \sum_{\Delta} M_{ijk} \sigma_i \sigma_j \sigma_k + L \sum_i \sigma_i$$

where the first sum is over all neighboring pairs, the second sum over all triangular faces of the lattice, and the last term represents the external magnetic field energy. Here in (1), we have allowed the possibility of nonuniform interactions.

The Ising model (1) is exactly soluble when only (anisotropic) pair interactions, or only (uniform) triplet interactions, are present. The general problem is very much intractable and has been studied in the past under various approximate schemes, including mean-field analysis, the Müller-Hartmann-Zittartz approximation, renormalization treatment, and Monte Carlo simulations. Recently, the case of uniform interactions has been considered by the use of the Yang-Lee zero consideration, and it was possible to deduce regimes in the parameter space that are free of phase transitions. In this paper we extend the consideration of Ref. 11, which holds more generally than that of Ref. 10, to include anisotropic pair and staggered triplet interactions.
2. Star-Triangle Transformation

Consider the lattice shown in Fig. 1 with (reduced) anisotropic pair interactions $K_1$, $K_2$, $K_3$, staggered triplet interactions $M$ and $M'$, and an external magnetic field $L$. As in Ref. 11, our procedure is to first convert the Ising lattice into one with only pair interactions and magnetic fields. This is done by the use of the star-triangle transformation. For this purpose, we write

$$K_i = J_i + J'_i$$

(2)

where

$$J_i = rK_i, \quad J'_i = (1 - r)K_i$$

$$r = M/(M + M')$$

(3)

and associate $J_i$ and $J'_i$ to the up-pointing and down-pointing triangular faces, respectively. We restrict our considerations to

$$|K_i| > |M + M'|$$

(4)

so that

$$|J_i| > |M|, \quad |J'_i| > |M'|$$

(5)

For each triangular face we introduce a star-triangle transformation which is shown in Fig. 2 (for the up-pointing triangles). We have the following relation between the Ising parameters:

![Fig. 1. The triangular lattice.](image)
Fig. 2. The star-triangle transformation for the up-pointing triangles.

\[ e^{M \sigma_1 \sigma_2 \sigma_3 + J_1 \sigma_1 \sigma_3 + J_2 \sigma_1 \sigma_2 + J_3 \sigma_2 \sigma_3} = e^{H_1 \sigma_1 + H_2 \sigma_2 + H_3 \sigma_3} \cosh(H + R_1 \sigma_1 + R_2 \sigma_2 + R_3 \sigma_3). \quad (6) \]

Explicitly, (6) gives rise to eight equations for the eight unknowns \( F, H, H_1, H_2, H_3, R_1, R_2, R_3 \). Writing them out, we have

\[ a = e^{M + J_1 + J_2 + J_3} - e^{H_1 + H_2 + H_3} \cosh(H + R_1 + R_2 + R_3) \]
\[ b_i = e^{-M + J_i - J_j - J_k} - e^{-H_i + H_j - H_k} \cosh(H - R_i + R_j + R_k) \]
\[ c_i = e^{M + J_i - J_j - J_k} - e^{H_i - H_j - H_k} \cosh(H + R_i - R_j - R_k) \]
\[ d = e^{-M + J_i + J_j + J_k} - e^{-H_i - H_j - H_k} \cosh(H - R_i - R_j - R_k) \quad (7) \]

with \( i, j, k = 1, 2, 3, i \neq j \neq k \neq i \). Equations (6) and (7) are the same as those considered, and solved, by us in a consideration of a general 8-vertex model on the honeycomb lattice.\textsuperscript{12} We now write down the solution in a slightly different form for later use:

\[ e^{4H_i} = \frac{b_i b_k - ac_i}{c c_k - db_i} = \frac{\sinh 2(J_i + M)}{\sinh 2(J_i - M)} \quad (8a) \]

\[ \cosh 2R_i = \frac{ad + b_i c_i - b_j c_j - b_k c_k}{2[(b_i b_k - ac_i)(c_i c_k - b_i d)]^{1/2}} \]
\[ = \frac{e^{2J_i} \cosh 2(J_i + J_k) - e^{-2J_i} \cosh 2(J_i - J_k)}{[2(\cosh 4J_i - \cosh 4M)]^{1/2}} \quad (8b) \]
\[
\sinh 2H = \sinh 2R_i \left( \frac{a_i e^{-2(H_j + H_k)} - c_i e^{2(H_j + H_k)}}{ad - b_i c_i} \right)
\]
\[
= - \sinh 2R_i \sinh (H_j + H_k) / \sinh (2J_j + J_k)
\]
\[
- \sinh 2R_i \sinh 4M
\]
\[
= \frac{1}{[(\cosh 2J_j - \cosh 2M)(\cosh 2J_k - \cosh 2M)]^{1/2}}.
\]

Here we have used (8a) in arriving at the last expression in (8c). Similarly, by carrying out a star-triangular transformation for the down-pointing triangles, we obtain
\[
e^{4H_i} = \frac{\sinh 2(J'_j + M')}{\sinh 2(J'_j - M')} \]
\[
\cosh 2R'_i = \frac{e^{2J'_j} \cosh 2(J'_j + J'_k) - e^{-2J'_j} \cosh 2(J'_j - J'_k)}{[2(\cosh 4J'_j - \cosh 4M')]^{1/2}} \]
\[
\sinh 2H' = - \sinh 2R'_i \sinh (H'_j + H'_k) / \sinh (2J'_j + J'_k)
\]
\[
- \sinh 2R'_i \sinh 4M'
\]
\[
= \frac{1}{[(\cosh 2J'_j - \cosh 2M')(\cosh 2J'_k - \cosh 2M')]^{1/2}}.
\]

These transformations lead to an equivalent diced lattice, which is shown in Fig. 3, with pure two-spin interactions \(R_i\) and \(R'_i\), magnetic fields \(H\) and \(H'\) at the 3-coordinated lattice sites, and a magnetic field
\[
L_0 = L + H_1 + H_2 + H_3 + H'_1 + H'_2 + H'_3
\]
at the 6-coordinated lattice sites. Note that the last expressions in (8c) and (9c) imply that, in the regime (4) and (5), all \(\sinh 2R_i\) (and \(\sinh 2R'_i\)) have the same sign, and that the signs of \(H\) and \(H'\) are given by
\[
\text{Sgn} (H) = - \text{Sgn} (\sinh 2R_i) \text{Sgn} (M)
\]
\[
\text{Sgn} (H') = - \text{Sgn} (\sinh 2R'_i) \text{Sgn} (M')
\]

This generalizes (8) of Ref. 11.

3. Ferromagnetic Pair Interactions and \(MM' > 0\)

Consider first the ferromagnetic regime \(K_i > 0\) and \(MM' > 0\), hence \(0 < r < 1\), \(J'_j > 0\), \(J_j > 0\). From (8b) and (9b) we see that both \(\cosh 2R_i\) and \(\cosh 2R'_i\) are
positive. In fact, by evaluating their minima at $M = 0$ and $M' = 0$, respectively, we establish the inequalities $\cosh 2R_i > 1$ and $\cosh 2R'_i > 1$. Therefore, without loss of generality, we may take both $R_i$ and $R'_i$ to be positive (ferromagnetic). Equation (11) then implies
\[ \text{Sgn}(H) = -\text{Sgn}(M), \quad \text{Sgn}(H') = -\text{Sgn}(M'), \] (12)
and, since $MM' > 0$, $H$ and $H'$ have the same sign. Now the Yang-Lee theorem\textsuperscript{1,3} states that, for ferromagnetic $R$ and $R'$, the Ising model exhibits no phase transition if its magnetic fields, namely $H$, $H'$, and $L_0$ in the present case, all have the same sign. Using the expression (10) for $L_0$, (8a), and (9a), and considering the cases $M > 0$, $M' > 0$, and $M < 0$, $M' < 0$ separately, we arrive at the conclusion that there is no phase transition in the regime
\[ \prod_{i=-1}^{3} \left( \frac{\sinh 2(J_i + |M|) \sinh 2(J'_i + |M'|)}{\sinh 2(J_i - |M|) \sinh 2(J'_i - |M'|)} \right) < e^{-4 \text{Sgn}(M)L}. \] (13)
This expression reduces to (2) of Ref. 11 in the case of uniform interactions $K_i = K > 0$, $M = M'$.

4. Antiferromagnetic Pair Interactions and $MM' > 0$

When one (or more) of the interactions $K_1$, $K_2$, and $K_3$ is negative, analysis can be carried out similarly. However, for the purpose of invoking the Yang-Lee
theorem and its extension, we again restrict considerations to (4) and $MM' > 0$, so that both $\cosh 2R_i$ and $\cosh 2R'_i$ are real. Furthermore, it is necessary to restrict to
\[
\cosh 2R_i < -1, \quad \cosh 2R'_i < -1
\] (14)
which is realizable when $K_i < 0$, including the isotropic case of $K_i = K < 0$ considered in Ref. 11. Explicitly, condition (14) can be written as
\[
\sinh^2 2M > \sinh^2 2J - [e^{2J} \cosh 2(J + J_i) - e^{-2J} \cosh 2(J_i - J_k)]^2 / 4, \quad (15a)
\]
\[
\sinh^2 2M' > \sinh^2 2J' - [e^{2J'} \cosh 2(J' + J'_i) - e^{-2J'} \cosh 2(J'_i - J'_k)]^2 / 4. \quad (15b)
\]
When (14) is satisfied, we have $2R_i = 2r_i \pm \imath \pi$ and $2R'_i = 2r'_i \pm \imath \pi$, where $r_i$ and $r'_i$ are real and they can be taken to be positive. It follow that we have $|e^{-2R_i}| = e^{-2r_i} < 1$, and $|e^{-2R'_i}| = e^{-2r'_i} < 1$ so that the generalization of the Yang-Lee theorem by Ruelle\textsuperscript{14} can be used. We have also
\[
\sinh 2R_i < 0, \quad \sinh 2R'_i < 0, \quad (16)
\]
and thus from (11) and $MM' > 0$, $H$ and $H'$ have the same sign as that of $M$ (or $M'$). The Yang-Lee-Ruelle theorem now says that there is no phase transition if $L_0$ also has the same sign. After using (10) for $L_0$ and considering the cases $M > 0, M' > 0$ and $M < 0, M' < 0$ separately, this leads to the regime
\[
\prod_{i=1}^{3} \left( \frac{\sinh 2(J_i + |M|)}{\sinh 2(J_i - |M|)} \frac{\sinh 2(J'_i + |M'|)}{\sinh 2(J'_i - |M'|)} \right) > e^{-4 \text{Sign}(M)L}. \quad (17)
\]
This expression reduces to (2) of Ref. 11 in the case of uniform interactions $K_i = K < 0$, $M = M'$.

5. The Regime $MM' < 0$

Considerations in the preceding sections can be extended when $M$ and $M'$ have opposite signs. Without loss of generality, we assume $M > 0, M' < 0$. Then there are two possibilities:

A. $(M + M')K_i > 0$

In this case we have always $J_i > 0$ and $J'_i < 0$. (For $K_i > 0, M + M' > 0$, we have $r > 1$ and for $K_i < 0, M + M' < 0$ we have $r < -1$.) Then, following discussions presented in the preceding two sections, we restrict to the regime $\cosh 2R'_i < -1$ or (15b), and we may choose $R_i$ and $R'_i$ such that we have
\[
\sinh 2R_1 > 0, \quad \sinh 2R'_1 < 0.
\]

It now follows from (11) that both \( H \) and \( H' \) are negative. Therefore, the Yang-Lee-Ruelle theorem dictates that there is no transition in \( L_0 \) negative, or,

\[
\prod_{i=1}^{3} \left( \frac{\sinh 2(J_i + M) \sinh 2(J'_i + M')}{\sinh 2(J_i - M) \sinh 2(J'_i - M')} \right) < e^{-4L}.
\]

(19)

B. \((M + M')K_i < 0\)

In this case we have always \( J_i < 0, J'_i > 0 \). (For \( K_i < 0, M + M' > 0 \), we have \( r > 1 \) and for \( K_i > 0, M + M' < 0 \) we have \( r < -1 \).) Then, we restrict to the regime (15a) and choose

\[
\sinh 2R_1 < 0, \quad \sinh 2R'_1 > 0
\]

so that both \( H \) and \( H' \) are positive. Therefore, there is no transition in the regime of positive \( L_0 \), or

\[
\prod_{i=1}^{3} \left( \frac{\sinh 2(J_i + M) \sinh 2(J'_i + M')}{\sinh 2(J_i - M) \sinh 2(J'_i - M')} \right) > e^{-4L}.
\]

(21)

5. Summary

We have considered the Ising model on the triangular lattice shown in Fig. 1, with anisotropic pair interactions \( K_1, K_2, K_3 \), nonuniform triplet interactions \( M, M' \), and an external magnetic field \( L \). We restrict to \( |K_i| > |M + M'| \).

For \( MM' > 0 \) and \( K_i > 0 \), we established that the Ising model is free of phase transitions in the regime (13). For \( MM' > 0 \) and in the regime (15a), (15b), which is realizable when \( K_i \) is negative, we established that the Ising model is free of phase transitions in the regime (17).

For \( M > 0 \) and \( M' < 0 \), we established that the Ising model is free of phase transitions in the regimes (15b), (19), \((M + M')K_i > 0\), and (15a), (21), \((M + M')K_i < 0\).

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References

13. See Lemma in Appendix II of Ref. 9.