Two-Dimensional Ising Model with Crossing and Four-Spin Interactions and a Magnetic Field $i(\pi/2)kT$

F. Y. Wu

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The Ising model on a checkerboard lattice with crossing and four-spin interactions is solved exactly when there is pure imaginary magnetic field $H = i(\pi/2)kT$. The model exhibits a critical point with continuously varying exponents.

KEY WORDS: Ising model; pure imaginary field; second-neighbor interactions; exact solution.

1. INTRODUCTION

The two-dimensional Ising model in a nonzero magnetic field is a well-known unsolved problem in statistical physics. In 1952 Lee and Yang\(^1\) obtained a solution for the two-dimensional nearest-neighbor model in the pure imaginary magnetic field

$$H = i\frac{\pi}{2}kT$$ (1)

where $T$ is the temperature. This solution, which has since been rederived from a variety of different approaches,\(^2\) exhibits a second-order phase transition occurring at infinite temperature. This leads to the occurrence of a zero-temperature phase transition in a fully frustrated Ising model\(^5,7\) in the dual space. It is also known that this solution of the nearest-neighbor model yields information on monomer correlations in the dimer problem.\(^2\)

In this paper we show that the phase transition occurring in the two-dimensional Ising model at the pure imaginary field (1) behaves differently

\(^1\) Department of Physics, Northeastern University, Boston, Massachusetts 02115.

when there are crossing and/or multispin interactions. We consider, and exactly solve, an Ising model with nearest-neighbor, next-nearest-neighbor, and four-spin interactions on a checkerboard-type lattice and in the presence of the magnetic field \(^{(1)}\).\(^{(8)}\) Our analysis shows that the model exhibits a phase transition occurring at finite temperatures. Furthermore, the critical exponents are continuous varying, i.e., they are dependent on the interactions. It is a curious fact that this Ising model, while unsolvable in zero magnetic field, becomes solvable in the presence of the pure imaginary field \(^{(1)}\).

2. DUALITY TRANSFORMATION

Consider an Ising model of \(N\) spins arranged on the square lattice as shown in Fig. 1. The four spins \(\sigma_1, \sigma_2, \sigma_3,\) and \(\sigma_4\) surrounding each shaded square in Fig. 1 interact with an energy

\[
E(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = -J_1(\sigma_1 \sigma_2 + \sigma_3 \sigma_4) - J_2(\sigma_2 \sigma_3 + \sigma_4 \sigma_1) - J_3 \sigma_1 \sigma_3 - J_4 \sigma_2 \sigma_4
\]

as indicated in Fig. 2. In addition, there is an external magnetic field \(H\) which we shall set at the fixed value \(^{(1)}\).

Fig. 1. The checkerboard Ising lattice.
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Fig. 2. Ising interactions (2) contained in a shaded square in Fig. 1. The four-spin interaction is not shown.

Since the thermodynamics of a system with complex Boltzmann factors may be boundary-condition-dependent, it is important to specify the precise boundary condition used. For our purposes we assume periodic boundary conditions. Write $L = H/kT$ and denote the partition function by $Z_N(L)$, where $L$ in general can be complex. Then, by using the identity

$$e^{i\pi\sigma^2} = i\sigma$$

the partition function of the Ising model can be written, at $L = i\pi/2$, as

$$Z_N\left(i\frac{\pi}{2}\right) = i^N \sum_{\sigma_i = \pm 1} \prod_{\text{shaded squares}} B(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$$

(4)

where

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_1 \sigma_2 \exp[-E(\sigma_1, \sigma_2, \sigma_3, \sigma_4)/kT]$$

(5)

is the Boltzmann factor associated with a shaded square in Fig. 1. The factor $i^N$ can be dropped if we assume $N$ to be multiples of 4.\(^3\)

Next we transform the partition function (4) into that of an Ising model with interactions in every square. This is a duality transformation which can be effected in a number of different ways.\(^{(9-11)}\) Here we follow a formulation due to Burkhardt,\(^{(10)}\) which also permits a discussion of the spin correlation function, by placing the $N/2$ dual spins $\mu_i$ in the unshaded squares (cf. Fig. 1). It is then straightforward by following the procedure

\(^3\)It can be quite easily verified that $Z_N(i\pi/2)$ is identically zero for $N = \text{odd}$. 

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given in Ref. 10 to rewrite the partition function (4) in the form of a spin summation in the dual space:

$$Z_N = \sum_{\mu_i = \pm 1} \prod_{\text{all squares}} W(\mu_1, \mu_2, \mu_3, \mu_4)$$  \hspace{1cm} (6)

where

$$W(\mu_1, \mu_2, \mu_3, \mu_4) = \frac{1}{4} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} (-1)^{\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1} \times B(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$$  \hspace{1cm} (7)

with

$$t_i = \frac{1}{2}(1 + \mu_i)$$
$$\sigma_{ij} = 1 - \delta_{kr}(\sigma_i, \sigma_j)$$

is the new "Boltzmann" weight for the dual Ising lattice. Note that this transformation is exact, and that the dual lattice has only $N/2$ spins and is oriented at a 45° rotation (cf. Fig. 1), also with periodic boundary conditions. It should also be noted that, when applied to the nearest-neighbor model, the duality transformation (7) corresponds to the decimation of half of the spins in the (fully frustrated) Ising model in the dual space, a procedure known to lead to an eight-vertex model at the decoupling point.\(^{(12)}\)

For Boltzmann weights $B(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ such as those given by (5) satisfying the spin-reversal symmetry, the weights $W(\mu_1, \mu_2, \mu_3, \mu_4)$ are also spin-reversal invariant. We can then write (7) explicitly as

$$W = X B$$  \hspace{1cm} (8)

where $W$ and $B$ are column vectors whose components are\(^{4}\)

$$W_1 = W(+ + + +), \quad B_1 = B(+ + + +)$$
$$W_2 = W(- + - +), \quad B_2 = B(- + - +)$$
$$W_3 = W(- - + +), \quad B_3 = B(- - + +)$$
$$W_4 = W(+ - - +), \quad B_4 = B(+ - - +)$$
$$W_5 = W(- - - +), \quad B_5 = B(- - - +)$$
$$W_6 = W(- - - -), \quad B_6 = B(- - - -)$$
$$W_7 = W(+ - - -), \quad B_7 = B(+ - - -)$$
$$W_8 = W(- - - -), \quad B_8 = B(- - - -)$$  \hspace{1cm} (9)

\(^{4}\)Note the reversal roles of $W_5, W_6$ and $B_5, B_6$ with respect to spin arguments.
and $\mathbf{X}$ is the symmetric $8 \times 8$ matrix

$$
\mathbf{X} = \frac{1}{2} \begin{bmatrix}
+ & + & + & + & + & + & + & + \\
+ & + & + & - & - & - & - & - \\
+ & + & - & - & - & + & + & + \\
+ & + & - & - & + & + & - & - \\
+ & - & + & + & - & + & + & - \\
+ & - & - & + & + & + & - & - \\
+ & - & + & - & - & - & + & + \\
+ & - & - & - & + & - & + & - \\
\end{bmatrix}
$$

Here $+(-)$ denotes $+1(-1)$. It can be easily verified that the inverse of (8) is

$$
\mathbf{B} = 2\mathbf{X}^T \mathbf{W}
$$

3. EQUIVALENCE WITH AN EIGHT-VERTEX MODEL

For weights $\mathbf{W}$ which are invariant under spin reversals $\mu_i \rightarrow -\mu_i$, it is possible to introduce a $(2 \times 1)$ mapping of the spin configurations into the arrow configurations of an eight-vertex model.\(^{(13,14)}\) This leads to the following exact equivalence:

$$
Z_N = Z_{8v}
$$

where $Z_{8v}$ is the partition function of an eight-vertex model in the dual space whose vertex weights are

$$
\omega_i = W_i, \quad i = 1, 2, \ldots, 8
$$

Here, we have adopted the usual convention in numbering the vertices in effecting the mapping.\(^{(15,16)}\)

It is now a simple matter to substitute (2) and (5) into (8), obtaining the following explicit expressions for the vertex weights:\(^5\)

$$
\{\omega_1, \omega_2, \ldots, \omega_8\} = \{a_+, a_-, b_+, b_-, c_+, c_-, d_+, d_-\}
$$

where

$$
a = a_+ = a_- = (uvt)^{-1}(\sinh x + u^2v^2 \sinh y)$$

$$
b = b_+ = b_- = (uvt)^{-1}(\sinh x - u^2v^2 \sinh y)$$

$$
c_\pm = (uvt)^{-1}[\cosh x + u^2v^2 \cosh y \mp t^2(u^2 + v^2)]$$

$$
d_\pm = (uvt)^{-1}[\cosh x - u^2v^2 \cosh y \mp t^2(u^2 - v^2)]
$$

\(^5\)If we have started with a checkerboard lattice with two different horizontal (and vertical) nearest-neighbor interactions $J_1, J_1$ (and $J_2, J_2$), then the resulting eight-vertex model has $a_+ \neq a_-, b_+ \neq b_-$, which has not been solved.
with
\[ x = 2(K_1 + K_2), \quad y = 2(K_1 - K_2) \]
\[ u = e^{-K}, \quad v = e^{-K'}, \quad t = e^{-K^4} \]
\[ K_i = J_i/kT, \quad K = J/kT, \quad K' = J'/kT \]

Since the vertices with weights \( \omega_5 \) and \( \omega_6 \), and those with weights \( \omega_7 \) and \( \omega_8 \), occur in pairs in the eight-vertex model, we may replace both \( c_+ \) by \( c \) and \( d_+ \) by \( d \) where
\[ c^2 = c_+ c_-, \quad d^2 = d_+ d_- \quad (16) \]

It follows that the partition function (4) is precisely that of an eight-vertex model with standard weights \( a, b, c, d \) given by (15) and (16).

Baxter\(^{(17)}\) has solved the eight-vertex model for real \( a, b, c, d \). Thus, the partition function (4) can be evaluated in the regime \( c_+ c_- > 0, d_+ d_- > 0 \), or, equivalently,
\[ |\cosh x + u^2 v^2 \sinh y| > t^2 |u^2 + v^2| \quad (17) \]

As is well known, the solution exhibits a transition with continuously varying exponents, occurring at the critical point
\[ |a| + |b| + |c| + |d| = 2\max\{|a|, |b|, |c|, |d|\} \quad (18) \]

### 4. FERROMAGNETIC MODEL

The above results are very general, applicable to ferromagnetic as well as antiferromagnetic interactions. For concreteness we now restrict ourselves to ferromagnetic interactions. It can be verified that for ferromagnetic interactions the vertex weights (16) are always positive and that the only possible realization of (18) is
\[ |c| = a + b + |d| \quad (19) \]

which, after some reduction, reduces to
\[ |\cosh y - t^4 \cosh x| = \sinh x [(u^{-4} - t^4)(v^{-4} - t^4)]^{1/2} \quad (20) \]

Here, without loss of generality, we have taken \( K_1 \geq K_2 \geq 0 \). Thus, the Ising model (2) is exactly solved at the fixed magnetic field (1) in the ferromagnetic regime. For pairwise interactions (\( K_4 = 0 \)) for which the
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Lee-Yang circle theorem\(^{(1)}\) is valid, the critical condition (20) reduces further to

\[
\coth 2K_1 + \coth 2K_2 = 2\left[ (e^{4K_1} - 1)(e^{4K_2} - 1) \right]^{-1/2}
\]  

We see that (21) yields a critical temperature \(T_c\) which is finite only when there are crossing interactions \((JJ' \neq 0)\). The critical temperature diverges when there is no crossing interaction \((JJ' = 0)\) and in the one-dimensional limit \(J_1 = J_2 = 0\).

5. CORRELATION FUNCTION

We can apply the duality transformation (7) to the two-spin correlation function \(\langle \sigma_{0,0} \sigma_{n,n} \rangle\), where \(\sigma_{i,j}\) is the spin located at the point \((i, j)\) in Fig. 1, to obtain an expression in the dual space. Writing

\[
\sigma_{0,0} \sigma_{n,n} = (\sigma_{0,0} \sigma_{1,1})(\sigma_{1,1} \sigma_{2,2}) \cdots (\sigma_{n-1,n-1} \sigma_{n,n})
\]  

and associating the factors \((\sigma_{x,x+1} \sigma_{x+1,x+1})\) to the appropriate shaded squares, we find

\[
\langle \sigma_{0,0} \sigma_{n,n} \rangle = Z_8^n/Z_8
\]  

where \(Z_8^n = Z(i\pi/2)\) is the partition function of the eight-vertex model whose vertex weights are given by (15), and \(Z_8^n\) is the partition function of the same eight-vertex model with vertex weights along a single row of \(n\) sites modified to new values. These new weights are obtained from (7) with the replacement

\[
B(o-1, 0-2, 0-3, 0-4) \rightarrow B_0(o-1, 0-2, 0-3, 0-4)
\]  

Inspection of (9) shows that this corresponds to negating \(B_3, B_4, B_7,\) and \(B_8\) which, by virtue of (7), leads to the interchanges

\[
W_1 \leftrightarrow W_4, \quad W_2 \leftrightarrow W_3, \quad W_5 \leftrightarrow W_8, \quad W_6 \leftrightarrow W_7
\]  

This further corresponds to the negation of the second-neighbor interactions in the spin representation (in the dual space) of the weights\(^{(13,14)}\).

Barber and Baxter\(^{(18)}\) have obtained the magnetization of the eight-vertex model considered in the spin language of the dual space, and it is of interest to understand their result in the context of the Ising model under consideration. This is done by applying the inverse transformation (11) to the spin correlation function \(\langle \mu_{0,0} \mu_{n,n} \rangle\). Again, writing

\[
\mu_{0,0} \mu_{n,n} = (\mu_{0,0} \mu_{1,1}) \cdots (\mu_{n-1,n-1} \mu_{n,n})
\]
and associating the \((\mu_{x,\alpha} \mu_{x+1,\alpha+1})\) factors to the corresponding shaded squares in effecting the transformation, we obtain
\[
\langle \mu_{0,0} \mu_{n,n} \rangle = Z_N^{(n)} \left( \frac{i \pi}{2} \right) / Z_N \left( \frac{i \pi}{2} \right)
\]
where \(Z_N(n/2)\) is the partition function given by (4), and \(Z_N^{(n)}(i \pi/2)\) is that of the same lattice but with signs of \(K, K'\) reversed in a row of \(n/2\) adjacent shaded squares. This also corresponds to the interchange of the Boltzmann weights
\[
B_1 \leftrightarrow B_3, \quad B_2 \leftrightarrow B_4, \quad B_5 \leftrightarrow B_7, \quad B_6 \leftrightarrow B_8
\]
for these \(n\) shaded squares. Barber and Baxter's evaluation of the magnetization indicates that the expression (27) vanishes identically above \(T_c\) in the \(n \to \infty\) limit.

6. LEE–YANG ZEROS

The Lee–Yang zeros are solutions of the equation
\[
Z_N(L) = 0
\]
in the complex \(z = e^{-2L}\) plane, which, for ferromagnets, lie on the unit circle. In the limit of \(N \to \infty\), the zeros attain a continuous distribution described by a density function \(g(\theta)\), where \(\theta\) is the azimuth angle of \(z\). Lee and Yang\(^{(1)}\) have shown that \(4\pi g(\theta)\) is precisely the amount of discontinuity of the magnetization
\[
I(L) = \frac{\partial}{\partial L} \lim_{N \to \infty} \frac{1}{N} \ln Z_N(L)
\]
across the unit circle for \(\theta\) fixed. Thus, the existence of a nonzero magnetization (and two-spin correlation function in the limit of an infinite separation) at \(\theta = \pi\) necessarily implies \(g(\pi) > 0\), as is found to be the case in the one-dimensional and the two-dimensional nearest-neighbor models.\(^{(1,2)}\) In both of these cases, we have \(T_c = \infty\). The situation is less clear when there are second-neighbor interactions, since \(T_c\) is now finite. Certainly we must have \(g(\pi) > 0\) below \(T_c\). For \(T > T_c\) it is tempting to suppose that a gap will occur in the distribution of zeros across the negative axis, as is in the case at the positive real axis.\(^{(1)}\) However, we know there is certainly one zero residing at \(\theta = \pi\) for \(N\) odd. While we cannot rule out the possibility that the zeros may actually possess a vanishing
density along an arc of the unit circle crossing the negative real axis, it appears more likely that the zeros are distributed continuously at all temperatures in the negative real half-plane. The function $g(\pi)$ will then exhibit some sort of singularity at $T_c$, perhaps vanishing above $T_c$. It would be useful to carry out numerical studies for large lattices to elucidate this point.

8. SUMMARY

We have obtained the exact solution of an Ising model with first-, second-, and four-spin interactions in the pure imaginary magnetic field $i\frac{\pi}{2}kT$. The solution exhibits a phase transition only when there are nonzero crossing and/or four-spin interactions, and the transition possesses continuously varying exponents. We also obtained an expression in the dual space for the spin–spin correlation function and discussed possible forms of the Lee–Yang zero distribution across the negative axis.

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