GRAPHICAL APPROACH TO THE NONINTERSECTING STRING MODEL: STAR–TRIANGLE EQUATION, INVERSION RELATION, AND EXACT SOLUTION

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A general q-state nonintersecting string model for $q \geq 2$ is studied, and solved, using a graphical approach. This model, of which special cases were first introduced by Stroganov and Schultz, has recently drawn further attention because of its relation to Potts models. In its simplest version, a uniform, separable nonintersecting string model on a quadratic lattice, it corresponds to a $q^2$-state Potts model with Potts-spins on every other lattice face.

We first formulate the related star–triangle equation which we solve in terms of a line-variable parametrization. Such a parametrization of a solution of the star–triangle equation leads to a simple and direct graphical derivation of an inversion relation for the partition function. Our graphical analysis also shows that the inversion relation holds if certain boundary effects can be neglected, as we shall give a special finite lattice from which the inversion relation can be read off immediately. This relation is next solved under appropriate analyticity assumptions, again using a simple and direct approach, and the result is applied to obtain exact solutions for specific string models.

1. Introduction

Since the early days of using graphs to represent terms in the high- and low-temperature expansions of an Ising partition function, graphical methods have proven to be an extremely powerful approach in the study of exactly solved problems in lattice statistics. Discussions of the basic concepts of graphical considerations as well as related derivations of important results can be found in a review by Kasteleyn\(^1\), one of the key investigators in this field. Another aspect of exactly solvable models is the connection with families of commuting transfer matrices, a property which can be further understood in terms of a local integrability condition or the star–triangle (Baxter–Yang) equation. Again, Kasteleyn\(^2\) has given a comprehensive review of this important topic.
These two treatises by Kasteleyn, on two seemingly unrelated topics, have provided for many newcomers to the field the inspiring introduction into the fascinating field of exactly soluble problems in lattice statistics.

In this paper we present an approach to exactly solvable problems which combines the two topics so eloquently described by Kasteleyn. By introducing a few new tricks of fairly general applicability, we shall show that the solution of a particular lattice statistical model can be obtained quite easily by applying graphical analysis to the star–triangle equation.

We consider the \( q \)-state nonintersecting string (NIS) model introduced by Perk and Schultz\(^3\,^4\). In a previous paper\(^5\), we considered a \( q \)-state NIS model with separable vertex weights on an arbitrary even-valenced planar lattice, and established its equivalence with a \( q^2 \)-state Potts model. Since Kasteleyn and Fortuin\(^6\,^7\) have shown that the Potts model describes many physical problems, the equivalence, which was also independently established by Truong\(^8\) for the uniform square lattice using a completely different approach, now highlights the importance of the NIS model. The equivalence with a Potts model also serves to deduce critical properties for NIS models on regular lattices from known results of the Potts model\(^9\,^{11}\), and vice versa.

In this paper we obtain an exact expression for the free energy of the NIS model when the star–triangle relation is satisfied. The derivation utilizes a graphical approach, a natural choice considering that the NIS model is graphically defined to begin with. Our approach also applies to oriented NIS models\(^5\) which provide new families satisfying the star–triangle relation. For all these NIS models the star–triangle condition corresponds to criticality in the related Potts model. Thus, as a corollary of our considerations, we have a simpler and more direct derivation of the free energy of the critical Potts model\(^9\,^{11}\). However, on a closer inspection of our arguments, we seem to be led to speculate about a much more startling amount of universality. There are by now many different solutions of the star–triangle equations (and we believe that many more are to be found). But, there seems to be room for only few distinct expressions for the resulting free energy, all others being linear combinations of these. All (or almost all) "irreducible" free energies may have been encountered already in the literature so far.

The present paper is organized as follows: The general NIS model is defined in section 2. In section 3 we first formulate the Baxter–Yang, or star–triangle, equation for a general \( q \)-state vertex model, and then solve it for the NIS model under consideration. We next discuss properties of this solution of the Baxter–Yang equation which we give a graphical interpretation. Then in section 4, we derive an inversion relation for the partition function in the case of a square lattice. This graphical derivation of the inversion relation, which is simple and direct, is new, and makes it possible to pinpoint assumptions made
in writing down the inversion relation. The inversion relation is next solved in section 5, again using an argument that simplifies the steps. The resulting exact solution of the partition function is specialized to specific models in section 6, where we also give some further conclusions and speculations.

2. *q*-state vertex model

Consider an arbitrary planar lattice, or graph, $\mathcal{L}$ of valence 4, and consider $\mathcal{L}$ as being formed from a set of coplanar lines, or curves, intersecting at $N$ points (sites). Each lattice edge of $\mathcal{L}$ can be in one of $q$ distinct states (colors). Then, a *q*-state vertex model on $\mathcal{L}$ is defined by specifying the vertex weight $w_i(\lambda, \mu, \alpha, \beta)$ at the $i$th site, where, as shown in fig. 1, $\lambda, \mu, \alpha, \beta = 1, \ldots, q$ denote the respective colors of the four incident edges. The problem confronting us is the evaluation of the per-site partition function defined by

$$\kappa = \lim_{N \to \infty} Z^{1/N},$$

with

$$Z = \sum_{\text{states}} \prod_{i=1}^{N} \omega_i(\lambda, \mu, \alpha, \beta)$$

for this *q*-state vertex model.

For the purpose of defining vertex weights, particularly in the case of nonregular lattices, a convention for fixing the relative positionings of the four incident edges is needed. We do this by orienting all lines in $\mathcal{L}$. Thus, there are always two oriented lines intersecting at a site. Our convention, shown in fig. 1, is to label the two edges with incoming arrows $\lambda, \alpha$ and the two edges with outgoing arrows $\mu, \beta$.

**Nonintersecting String (NIS) model**

The *q*-state vertex model with the most general vertex weights $\omega_i(\lambda, \mu, \alpha, \beta)$ is a $q^4$-vertex model and has not been solved. The nonintersecting string (NIS) model is a subclass of the general *q*-state model in which lattice edges form nonintersecting polygons of different colors. This is also a $q(2q-1)$-vertex model.

* Such a convention is not needed in ref. 5 since the vertex weights there can be assigned on the basis of a bipartite consideration.
In ref. 5 we considered a particular class of the NIS model, namely, that with separable vertex weights. We now consider a more general case with the nonvanishing weights

\[ \omega_i(\rho, \rho, \rho, \rho) = (W_{\rho\rho})_i = c_{ip}^2, \]
\[ \omega_i(\rho, \sigma, \rho, \sigma) = (W_{\rho\sigma})_i = a_{ip}a_{iso}, \quad i = 1, \ldots, N, \rho \neq \sigma, \]
\[ \omega_i(\sigma, \rho, \rho, \sigma) = (W_{\sigma\rho})_i = b_{ip}b_{iso}, \]

where we have allowed the vertex weights to be color and site dependent. Further, motivated by results of previous considerations\(^{4,12,13}\), we let

\[ \{a_{ip}^2, b_{ip}^2, c_{ip}^2\} = \begin{cases} N_i\{s_i, 1, s_i + 1\}, & \rho = 1, 2, \ldots, n, \\ N_i\left\{s_i\left[\frac{s_i + e^{-\eta}}{s_i + e^{\eta}}\right]^{1/2}, \left[\frac{1 + s_i e^{-\eta}}{1 + s_i e^{\eta}}\right]^{1/2}\right\}, & \rho = n + 1, \ldots, q, \end{cases} \]

where\(^*\)

\[ e^\eta = \frac{1}{2}\left[n + \sqrt{n^2 + 4(q - n - 1)}\right] \]

and \(s_i, N_i\) are (real or complex) parameters which are site dependent and color independent. In this paper we shall assume \(\eta\) to be positive or \(q > 2\)^\(^{**}\). Note that the replacement of \(s_i \rightarrow s_i^{-1}\) (and \(N_i \rightarrow N_is_i\)) in (4) is equivalent to the interchange of \(a_{ip}^2\) and \(b_{ip}^2\). This is an important property that we shall later refer to.

\(^*\) The quantity \(e^\eta\) is called \(K\) in refs. 12 and 13, and \(K^{-1}\) in refs. 3 and 4.

\(^{**}\) The case \(q = 2\) is the six-vertex subcase of the eight-vertex model, see refs. 2 and 9 for review, with \(b = 0\) and \(a = c + d\); this case is included in the following as the limit \(\eta \downarrow 0\). We note that for \(q = 2\) eq. (5) gives \(\eta = 0\) for all three choices of \(n\) \((n = 0, 1, 2)\).
For fixed \( q \) and choosing

\[ n = 0, 1, \ldots, q, \]

(4) defines \( q + 1 \) distinct NIS models. Note that only the \( n = 0 \) and \( n = q \) models have color-independent vertex weights, and that the \( n = q \) model is the separable model considered in ref. 5. The \( n = 0 \) and \( n = q \) models are the two cases considered previously by Schultz\(^{14}\), and in these two cases the weights are restricted by

\[ c_i^2 = a_i^2 + b_i^2 \quad (n = q), \]

\[ c_i^4 = c_i^2(a_i^2 + b_i^2) + (q - 2)a_i^2b_i^2 \quad (n = 0). \]

The \( q(2q - 1) \)-vertex model defined by (4) is still very general and has not yet been solved. We shall, however, obtain a condition under which, for regular lattices at least, the per-site partition function can be evaluated. This leads us to consider the star–triangle equation.

3. Star–triangle equation

We first consider the general \( q \)-state vertex model whose partition function is defined by (2). Our procedure is to allow lines of \( L \) to be shifted, passing through lattice points. The star–triangle equation is then the condition under which the partition function remains unchanged under this shifting operation. Baxter\(^{15}\) has introduced this condition as the star–triangle relation for the \( Z \)-invariant model, a condition which is usually derived from a consideration of commuting transfer matrices\(^{16}\), and is equivalent to the factorizability condition of the \( S \)-matrix for particle scattering in one dimension\(^{17,18}\)). Here we write down the star–triangle equation for the most general \( q \)-state vertex model and its solution for the NIS model (4) and the oriented NIS model of ref. 5.

3.1. General \( q \)-state vertex model

Consider the shifting operation shown in fig. 2 when a line is shifted passing a lattice point. The operation involves the change of the relative positionings of three lattice points, numbered 1, 2, and 3, in fig. 2. The three sites are connected to the rest of the lattice through six edges whose state (color) labels \( \alpha_1, \ldots, \alpha_6 \) remain unchanged. It follows that the overall partition function will
remain unchanged provided that the vertex weights \( \omega_1, \omega_2, \omega_3 \) at the three sites satisfy the relation

\[
\sum_{\beta_1, \beta_2, \beta_3 = 1}^q \omega_1(\beta_2, \alpha_2, \beta_1, \alpha_1)\omega_2(\alpha_6, \beta_3, \alpha_5, \beta_2)\omega_3(\beta_3, \alpha_3, \alpha_4, \beta_1) = \sum_{\beta_1, \beta_2, \beta_3 = 1}^q \omega_1(\alpha_5, \alpha_2, \beta_1, \alpha_1)\omega_2(\beta_3, \alpha_3, \beta_2, \alpha_2)\omega_3(\alpha_6, \beta_3, \beta_1, \alpha_1). \tag{8}
\]

This is the star–triangle equation for the general \( q \)-state vertex model. In the most general case (8) is a set of \( q^6 \) equations involving \( 3q^4 \) vertex weights and, obviously, a set of over-determined equations. A major task in the study of lattice models is to determine whether (8) possesses a solution and, if it does possess a solution, whether the partition function can be evaluated under this restriction.

### 3.2. Nonintersecting string model

The most general solution of (8) is not yet known. However, Schultz\(^{14}\) and Perk and Schultz\(^{12,13}\) have analyzed the star–triangle equation (8) for the NIS model (3), obtaining a solution for the vertex weights given by (4) and (5). However, details of their analysis have not been published. It, therefore, appears desirable to present here the analysis in some simpler cases. Here, we use a graphical approach to solve (8) for the separable \((n = q)\) NIS model considered in ref. 5.

The vertex weights of the separable NIS model are given by (3) with color-independent parameters \( a_i, b_i, c_i \), subject to \( c_i^2 = a_i^2 + b_i^2 \), as given in (7). This permits us to write

\[
\omega_i(\lambda, \mu, \alpha, \beta) = \delta_{\lambda \alpha} \delta_{\mu \beta} a_i^2 + \delta_{\lambda \beta} \delta_{\alpha \mu} b_i^2. \tag{9}
\]
Next we substitute (9) into (8). The substitution is facilitated by the use of a graphical representation similar to that used in fig. 3 of ref. 5. Representing each term in (8) by a diagram in which edges in the same state are connected, we obtain the graphical representation of (8) shown in fig. 3. Terms in (8) can be read off from fig. 3. For example, the first diagram in fig. 3 represents the term

$$q\delta_{12}\delta_{34}\delta_{56}a_1^2a_2^2b_3^2,$$

where $\delta_{ij} = \delta_{\alpha_1, \alpha_2}$, etc., and the factor $q$ comes from the internal loop which can be in any of the $q$ colors. In addition, each vertex carries a weight $b_1^2$ ($a_1^2$) if the four arrows at the vertex form (do not form) oriented flows.

It is now straightforward to equate corresponding terms on both sides of (8) according to the way the six external edges are connected. These equalities can be easily read off from fig. 3, yielding,
Writing

\[ s_i = a_i^2/b_i^2, \]

which holds for the \( n = q \) model as seen from the first line of (4), (10) leads to a single equation

\[ q + s_1^{-1} + s_2^{-1} + s_3 = s_1^{-1}s_2^{-1}s_3. \]  

(12)

This is the star–triangle equation for the separable \((n = q)\) NIS model. Note that (10), hence (12), is valid for \( q > 2 \), and that (12) is symmetric in \( s_1^{-1}, s_2^{-1}, \) and \( s_3 \), a consequence of the relative line-orientations imposed (cf. fig. 2).*

We remark at this point that the star–triangle equation can be solved in the same way for the oriented separable NIS model whose vertex weights are given by fig. 4 of ref. 5. This leads to the same solution (12), with \( q \) given by eq. (12) of ref. 5.

For the general \((n \neq q)\) NIS model the solution of the star–triangle equation is more complicated and we refer the readers to ref. 3 for details. The result is quite simple, however, and can be verified in a straightforward fashion. It suffices to state here that the vertex weights given by (3) and (4) satisfy the star–triangle equation (8) provided that, for \( q > 2 \), we have

\[ 2 \cosh \eta + s_1^{-1} + s_2^{-1} + s_3 = s_1^{-1}s_2^{-1}s_3. \]

(13)

Expression (13) reduces to (12) in the separable case by taking \( n = q \) in (5). From here on we shall consider the star–triangle equation (13) valid for all \( n \) (and the more general oriented case).

* Eqs. (10)–(13) can be made fully symmetric in the indices 1, 2, 3 if we interchange \( a_3 \) and \( b_3 \), which, as noted before, changes \( s_3 \to s_3^{-1} \). This corresponds to a rotation of the vertex 3.
3.3. Parametrization of (13)

We now introduce a graphical interpretation of the star–triangle equation (11), which will facilitate our derivation of the inversion relation. The idea is to parametrize the vertex weights such that the star–triangle equation becomes an identity. This can be achieved by associating a line variable* to each line in \( \mathcal{L} \) and require the weight of a vertex to depend only on the difference of the line variables of the two lines intersecting at the vertex. Since only two of the (three) differences formed from a set of three variables are independent, this parametrization of the vertex weights implies that a relation exists between the three vertex weights in (8). The goal is then to choose a parametrization such that this relation is precisely the star–triangle equation (13).

Let \( u, v, w \) be the line variables of the three lines forming the vertices 1, 2, 3 as shown in fig. 2. It is then readily verified that (13) is an identity if we write

\[
\begin{align*}
  s_1 &= \frac{\sinh(u - v)}{\sinh(\eta - u + v)}, \\
  s_2 &= \frac{\sinh(v - w)}{\sinh(\eta - u + w)}, \\
  s_3 &= \frac{\sinh(u - w)}{\sinh(\eta - u + w)}.
\end{align*}
\]

We can summarize (14) by parametrizing all vertex weights by

\[
s_i = \frac{\sinh \tilde{U}/\sinh(\eta - \tilde{u})},
\]

where \( \tilde{u} \) is the difference of the two line parameters. For the situation shown in fig. 1, we have \( \tilde{u} = v - u \). Substitution of (15) into (4) then leads to the following expression:

\[
\{a_\rho^2, b_\rho^2, c_\rho^2\} = \begin{cases} \frac{1}{\sinh \eta} \{\sinh \tilde{u}, \sinh(\eta - \tilde{u}), \sinh \tilde{u} + \sinh(\eta - \tilde{u})\}, & \rho = 1, 2, \ldots, n, \\ \\
\frac{1}{\sinh \eta} \{e^{\eta \tilde{u}} \sinh \tilde{u}, e^{\eta \sinh(\eta - \tilde{u})}, \sinh \eta\}, & \rho = n + 1, \ldots, q. \end{cases}
\]

Here we have taken \( N_i = \sinh(\eta - \tilde{u})/\sinh \eta \) in (4) and dropped the index \( i \) as it is now redundant. (The vertex weight is now fully determined by the

* The concept of the line variables, or line parameters, was first introduced by Baxter in ref. 15, and is closely related to the concept of rapidities in field theories in 1+1 dimensions with factorizable S-matrices. See, e.g., a review by Zamolodchikov (ref. 19).
difference of the two line variables.) It is worthy to be pointed out that the difference between (16) and (4) is that vertex weights given by (16) satisfy the star–triangle equation automatically, while those given by (4) do not.

Since the vertex weights now depend only on \( \bar{u} = v - u \), we may write

\[
\omega_i(\lambda, \mu, \alpha, \beta) = R_{\alpha\beta}^{\lambda\mu}(v - u) .
\]  

(17)

The symbol \( R \) now reminds us that the weights satisfy the star–triangle equation. It follows from (3) that we have, explicitly,

\[
R_{\alpha\beta}^{\lambda\mu}(v - u) = \begin{cases} 
    c^2_{\alpha}, & \text{if } \lambda = \mu = \alpha = \beta , \\
    a_{\alpha}a_{\beta}, & \text{if } \lambda = \alpha \neq \beta = \mu , \\
    b_{\alpha}b_{\beta}, & \text{if } \mu = \alpha \neq \beta = \lambda . 
\end{cases}
\]

(18)

where \( \{a^2_{\alpha}, b^2_{\alpha}, c^2_{\alpha}\} \) are given by (16). Particularly we have

\[
\{a^2_{\alpha}, b^2_{\alpha}, c^2_{\alpha}\} = \{0, 1, 1\} , \text{ for } \bar{u} = 0 ,
\]

(19)

implying that

\[
R_{\alpha\beta}^{\lambda\mu}(0) = \delta_{\alpha\mu} \delta_{\beta\lambda} .
\]

(20)

The relation (20) indicates that two lines intersecting with equal line variables are, in effect, decoupled. This decoupling, which proves useful in later considerations, is shown graphically in fig. 4.

3.4. Unitarity relation

The solution (18) of the star–triangle equation (8) possesses a further decoupling property which we now state as a lemma:

Lemma (Unitarity).

\[
\sum_{\gamma_1 = 1}^{q} \sum_{\gamma_2 = 1}^{q} R_{\alpha_1\gamma_1}^{\alpha_2\gamma_2}(u - v)R_{\gamma_1\gamma_2}^{\beta_1\beta_2}(v - u) = C(v - u)\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} ,
\]

(21)

where

\[
C(v - u) = \frac{\sinh(\eta + u - v)\sinh(\eta + v - u)}{\sinh^2 \eta} = C(u - v) .
\]

(22)

The lemma can be given a graphical interpretation as shown in fig. 5, which shows that two lines forming two vertices in succession are decoupled, leaving
Fig. 4. The decoupling of two lines with equal line variables as indicated by (20).

Fig. 5. Graphical representation of the lemma, indicating unitarity of the solution of the star–triangle equation.

an overall constant $C(v - u)$. The value of $C(v - u)$ depends, of course, on the choice of the constants $N_i$ in (4), and would have been 1 if we had chosen $N_i = 1$ to begin with. For this reason (21) is often referred to as the unitarity condition, introduced in the context of particle scattering in $1 + 1$ dimensions$^{19}$).

The lemma can be established by straightforwardly substituting (18) into (21). However, it is more convenient to prove it using (20), for which the “hard” work of substitution has already been done. In fact, this is most conveniently done graphically. Using the graphical interpretation of (20) as given by fig. 4 and considering a configuration involving eight external edges shown in fig. 6, we can move the lines one at a time according to fig. 4. These steps are given in fig. 6. If we denote the LHS of (21) by $T_{\alpha_1\beta_2}(v - u)$, then the equality of the first and last diagrams in fig. 6 leads to

\[
\delta_{\alpha_1,\beta_1} \delta_{\alpha_2,\beta_2} T_{\alpha_3\beta_3}(v - u) = T_{\alpha_1\alpha_2}(v - u) \delta_{\alpha_3,\beta_3} \delta_{\alpha_4,\beta_4},
\]

which implies that
where $C$ is a constant. To determine $C$ we consider the special case of $\alpha_1 = \beta_1 = \rho$, $\alpha_2 = \beta_2 = \sigma$ and represent (24) by fig. 7. We find

$$C = b_{\rho}(v - u)b_{\rho}(u - v)b_{\sigma}(v - u)b_{\sigma}(u - v).$$

(25)

This leads to (22) upon using (16).

3.5. Symmetry of $R^{L\mu}_{\alpha\beta}(v - u)$

The solution (16) of the star–triangle equation possesses the following symmetry:

$$R^{L\mu}_{\alpha\beta}(\bar{\eta}) = R^{L\mu}_{\beta\alpha}(\bar{\eta}) = R^{L\mu}_{\mu\lambda}(\bar{\eta}) = R^{L\mu}_{\lambda\mu}(\bar{\eta}) = R^{L\mu}_{\alpha\beta}(\eta - \bar{\eta}) = R^{L\mu}_{\beta\alpha}(\eta - \bar{\eta}) = R^{L\mu}_{\mu\lambda}(\eta - \bar{\eta}) = R^{L\mu}_{\lambda\mu}(\eta - \bar{\eta}).$$

(26)
Fig. 7. Determination of the constant C in (24).

The first line of (26) can be read off directly from (18), and the second line of (26) is obtained by noting from (16) and (18) that the interchange of $a_p \leftrightarrow b_p$ corresponds simply to the exchanges of $\tilde{u} \leftrightarrow \eta - \tilde{u}$ and $\mu \leftrightarrow \lambda$ or $\alpha \leftrightarrow \beta$.

Remark. These symmetry relations also have a graphical interpretation\textsuperscript{15,19). If we introduce

$$\phi = \pi(v - u)/\eta$$

as the angle between the two edges with outgoing arrows, then the symmetry relation (26) can be interpreted as the invariance of the vertex weight $\omega_i(\lambda, \mu, \alpha, \beta)$ under the four rotations and the four reflections of the indices $\lambda, \mu, \alpha, \omega$ provided that we keep the intersecting lines straight at their intersection. This situation is shown in fig. 8. Note that now the arrows need not to be shown since the specification of the angle $\phi$ or $\pi - \phi$ is sufficient to determine the vertex weight and the relative edge orientations.

3.6. Formulation of soluble models

In the next two sections we shall show that the $q$-state vertex models whose vertex weights are parametrized by line variables are exactly soluble. Therefore, it is convenient to reformulate vertex models in terms of vertex weights given by (17), (18) and (16) where the relevant parameter is $\tilde{u}$. Now the parameter $\tilde{u}$ of the sites surrounding a face of the lattice satisfy a sum rule. This means that, after using (15) or $s_i = \sinh \tilde{u}/\sinh(\eta - \tilde{u})$, the parameters $s_i$ of the lattice are constrained, and this constraint becomes the solubility condition for the vertex model (4).
Thus, starting from any $q$-state vertex model on a planar lattice whose vertex weights are given by (4), we can construct a soluble model by introducing the line variable parametrization (15). This leads to, in turn, a solubility condition for the vertex weights (4).

4. Inversion relation (square lattice)

The partition function of soluble spin models often satisfies an "inversion relation" which can be derived in many different ways. In this section we introduce a graphical approach to derive the inversion relation for the NIS model (18). The method we adopt is simpler than the usual algebraic approach; it also takes into account the boundary conditions precisely, making it possible to identify the approximations, if any, involved in deriving the inversion relation.

Our derivation of the inversion relation is based on successive applications of the unitarity lemma established in subsection 3.4. Consider a planar lattice $\mathcal{L}^2$ which is made of $M$ concentric directed loops of line variables $u_1, u_2, \ldots, u_M$ and $N$ concentric directed loops of line variables $v_1, v_2, \ldots, v_N$, as shown in fig. 9. The lattice therefore consists of precisely $2MN$ vertices and $4MN$ edges. Next consider a NIS model defined on $\mathcal{L}^2$ with vertex weights now given by (18) and the line-variable parametrization (16) satisfying the star–triangle equation. It is clear by inspecting fig. 9 that we can apply the lemma exactly $MN$ times to decouple all the $M + N$ loops. Since the trace over unity, the weight of a closed loop, gives rise to a factor $q$, the partition function of the NIS model (18) on $\mathcal{L}^2$ can now be evaluated exactly, leading to the expression

$$Z_{\mathcal{L}^2} = q^{M+N} \prod_{i=1}^{M} \prod_{j=1}^{N} C(v_j - u_i).$$

(28)
Consider next a NIS model defined on an $M \times N$ lattice which contains $MN$ lattice points in the upper half of the $L^2$ lattice. Let its partition function be

$$Z_{M,N}(u_1, \ldots, u_M; v_1, \ldots, v_N) = Z_{M,N}(u; v)$$

under a specific (free or periodic) boundary condition. Similarly, let

$$Z_{N,M}(v_N, \ldots, v_1; u_M, \ldots, u_1) = Z_{N,M}(v; u)$$

be the partition function of a NIS model on an $N \times M$ lattice consisting of $NM$ points contained in the bottom half of $L^2$, again with a given boundary condition. It is then clear that $Z_{M,N}$ differs from the product $Z_{M,N}(u; v)Z_{N,M}(v; u)$ only in the Boltzmann factors involving the boundary edges. That is, while the evaluation of $Z_{M,N}(u; v)$ and $Z_{N,M}(v; u)$ individually requires the imposition of individual boundary conditions for each half of the lattice $L^2$, the evaluation of $Z_{x^2}$, which is exactly done in (28), requires the boundaries of the two halves of $L^2$ to be connected exactly as shown in fig. 9.

If all Boltzmann factors are nonnegative, then since there are more terms in the product $Z_{M,N}(u; v)Z_{N,M}(v; u)$ than in $Z_{x^2}$, we have the inequality

$$Z_{M,N}(u; v)Z_{N,M}(v; u) \geq Z_{x^2}.$$  \hspace{1cm} (29)

In the case of spin models, the inequality (29) and the condition on the positivity of the Boltzmann factors are sufficient to establish that the equality
actually holds in the thermodynamic limit\cite{23}). The proof of the similar result for vertex models is more subtle and has been studied for the $q = 2$ models only\cite{24}). Here, we shall assume that $Z_{M, N}(u; v) Z_{N, M}(v; u)$ are equal in the thermodynamic limit. Thus, we have

$$\kappa(u; v)\kappa(v; u) = \lim_{M, N \to \infty} \left[ \prod_{i=1}^{M} \prod_{j=1}^{N} C(v_j - u_i) \right]^{1/MN},$$

(30)

where $\kappa = \lim_{M, N \to \infty} Z^{1/MN}$ is the partition function per site of the $M \times N$ lattice. Expression (30) is the inversion relation for the NIS model.

In the uniform case

$$u_i = u, \quad i = 1, \ldots, M,$$
$$v_j = v, \quad j = 1, \ldots, N,$$

the inversion relation (30) becomes

$$\kappa(\bar{u})\kappa(-\bar{u}) = C(\bar{u}),$$

(31)

where $\bar{u} = v - u$ and $C(\bar{u}) = C(-\bar{u})$ is given by (22). This result is in agreement with that obtained by Schultz\cite{30,32}.

In the next section we shall solve (31), together with appropriate analyticity assumptions, for $\kappa(v - u)$. Then, following an argument due to Baxter\cite{9}, the partition function per site for the case of the square lattice with nonuniform weights can be obtained simply from

$$\kappa_{\text{nonuniform}} = \lim_{M, N \to \infty} \left[ \prod_{i=1}^{M} \prod_{j=1}^{N} \kappa(v_j - u_i) \right]^{1/MN}.$$  

(32)

The solution can be extended to any lattice made of intersecting lines by, again, following Baxter’s arguments\cite{9}. The per-site partition function of the general $Z$-invariant NIS model is

$$\kappa = \lim_{N \to \infty} \left[ \prod_{\langle i, j \rangle} \kappa(u_i - u_j) \right]^{1/N},$$

(33)

where $N$ is the total number of vertices and the product is taken over all pairs of intersecting lines $i$ and $j$ with line variables $u_i$ and $u_j$.

The validity of the inversion relation (31) appears to be justified only by the success in using it to obtain the per-site partition function in many known
cases. We wish to emphasize that a rigorous proof of the validity of (31) is still an open question. Indeed, it is known that, when there exist negative vertex weights such as in the antiferromagnetic Potts model\cite{23}, the per-site partition function is boundary-condition dependent, and that (31) certainly does not hold. Our analysis now identifies the approximations involved in writing (31) and makes it possible to further study conditions under which the inversion relation is valid.

5. Solution of the inversion relation (square lattice)

In the previous section we saw that, using (32), the per-site partition function of a general inhomogeneous NIS model can be constructed from that of a NIS model with uniform weights. It is therefore only necessary to evaluate \( \kappa(v - u) \) for the uniform model. We now proceed to carry out this evaluation. It must be pointed out at this point that the procedure of computing the per-site partition function using the inversion relation has previously been discussed by a number of authors\cite{3,19,20,22,26,27}. However, for the sake of completeness as well as for making explicit the assumptions, it is worthwhile to present here details of some crucial steps. The route we take to obtain the exact solution is also somewhat simpler than that given in prior discussions.

The left-right (and up-down) symmetry given by (26) implies that the per-site partition function satisfies the symmetry

\[
\kappa(\eta - \tilde{u}) = \kappa(\tilde{u}) ,
\]

where \( \tilde{u} = v - u \). Furthermore, replacing \( \tilde{u} \) by \( \tilde{u} + i\pi \) in (16) changes \( \{a_\rho, b_\rho, c_\rho\} \rightarrow i \{a_\rho, b_\rho, c_\rho\} \) and the factor \( i \) can be dropped since the number of corners, hence \( a_\rho, b_\rho, c_\rho \) factors, contained in each closed loop is always a multiple of 4. Thus, we have the further symmetry

\[
\kappa(\tilde{u}) = \kappa(\tilde{u} + i\pi) .
\]

We now show that, regarding \( \tilde{u} \) as a complex variable, relations (31), (34) and (35), together with appropriate analyticity assumptions, are sufficient to uniquely determine the function \( \kappa(\tilde{u}) \).

First we establish the fact that there can exist only one solution which satisfies (31), (34) and (35). To see this, we assume the contrary letting there exist two solutions and denote their ratio by \( r(\tilde{u}) \). It then follows from (31), (34) and (35) that
\[ r(\tilde{u})r(-\tilde{u}) = 1, \quad (36a) \]
\[ r(\eta - \tilde{u}) = r(\tilde{u}), \quad (36b) \]
\[ r(\tilde{u} + i\pi) - r(\tilde{u}). \quad (36c) \]

Combining (36a) with (36b) we obtain
\[ r(\tilde{u} + 2\eta) = \frac{1}{r(\tilde{u} + \eta)} = r(\tilde{u}). \quad (37) \]

Relations (36c) and (37), where \( \eta \) is real, indicate that \( r(\tilde{u}) \) is a function doubly periodic in the entire \( \tilde{u} \)-plane.

Now we make the crucial assumption* that \( \kappa(\tilde{u}) \) is analytic and free of zeros in the region
\[ 0 < \text{Re} \tilde{u} \leq \eta, \]
\[ 0 \leq \text{Im} \tilde{u} < \pi. \quad (38) \]

Then the double periodicity implies that \( r(\tilde{u}) \) is analytic (and free of zeros) in the entire \( \tilde{u} \)-plane and thus, by Liouville’s theorem, \( r(\tilde{u}) \) is a constant. From (36a) and the fact that \( \kappa > 0 \) by definition for positive weights, we now obtain
\[ r(\tilde{u}) = 1. \quad (39) \]

It follows that the two solutions are identical and we conclude that there exists only one solution which satisfies (31), (34), (35) under the analyticity assumption (38).

We now proceed to construct this solution. Rewrite (31), after using (34) and (22), as
\[ \kappa(\tilde{u})\kappa(\tilde{u} + \eta) = C(\tilde{u}) = \frac{(1 - e^{2\tilde{u}-2\eta})(1 - e^{-2\tilde{u}-2\eta})}{(1 - e^{-2\eta})^2}. \quad (40) \]

The particular form of (40) suggests that we look for a solution of the form
\[ \kappa(\tilde{u}) = g(\tilde{u})h(\tilde{u})/(1 - e^{-2\eta}), \quad (41) \]
where \( g(\tilde{u}) \) and \( h(\tilde{u}) \) satisfy

* Similar assumptions have been made and discussed in refs. 9, 19, 22 and 27.
\[ g(\bar{u})g(\bar{u} + \eta) = 1 - e^{2\bar{u} - 2\eta} , \quad (42) \]
\[ h(\bar{u})h(\bar{u} + \eta) = 1 - e^{-2\bar{u} - 2\eta} , \quad (43) \]

and are analytic and free of zeros in the region (38). We can iterate (43) to obtain
\[
h(\bar{u}) = \frac{1 - e^{-2\bar{u} - 2\eta}}{h(\bar{u} + \eta)} = \frac{1 - e^{-2\bar{u} - 2\eta}}{1 - e^{-2\bar{u} - 4\eta}} h(\bar{u} + 2\eta)
= H(\bar{u}) \prod_{n=1}^{\infty} \left[ \frac{1 - e^{-2\bar{u} - 2(2n-1)\eta}}{1 - e^{-2\bar{u} - 4n\eta}} \right] , \quad (44)\]

where the function \( H(\bar{u}) \) is yet undetermined.

We shall set
\[
H(\bar{u}) = 1 . \quad (45)\]

Comparing (42) with (43) we see that (42) is solved by
\[
g(\bar{u}) = h(\eta - \bar{u}) . \quad (46)\]

Combining (41) and (44)-(46), we finally obtain
\[
\kappa(\bar{u}) = \frac{1}{1 - x} \prod_{n=1}^{\infty} \left[ \frac{(1 - x^{2n} e^{2\bar{u}})(1 - x^{2n-1} e^{-2\bar{u}})}{(1 - x^{2n+1} e^{2\bar{u}})(1 - x^{2n} e^{-2\bar{u}})} \right] , \quad (47)\]

where \( x = e^{-2\eta} \). It is easily checked that (47) satisfies the inversion relation (31), the symmetry relations (34) and (35), and is analytic and free of zeros in (38). Therefore, from the argument we have previously given, (47) is the solution. This infinite product expression agrees with the solution of the NIS model given previously\(^{3,13,20}\). Expression (47) also gives the solution of the oriented NIS model of ref. 5.

6. Applications and discussions

In the previous sections we have obtained in a fairly simple way the partition function per site for a large class of NIS models with an underlying line variable description. We now apply the result to specific models and remark on its further generalizations. We conclude with some remarks on inversion relations.
6.1. Exact solution for Potts models

It was shown in ref. 5 (see also ref. 8) that the separable \( n = q \) \( q \)-state NIS model is equivalent to a related \( q^2 \)-state Potts model. For \( n = q \) the parameter \( \eta \) defined by (5) is related to \( q \) through the simple relation

\[
2 \cosh \eta = q = Q^{1/2}.
\]

We may now treat \( \eta \), hence \( q \) and \( Q \), as a continuous variable, and our results are applicable to a \( Q \)-state Potts model where \( Q \geq 4 \) is not necessarily the square of an integer. The \( Q \)-state Potts model generally possesses arbitrary nearest neighbor interactions which may be site dependent. The line variable parametrization then leads to a subspace of the parameter space in which the Potts model is soluble.

Consider, for definiteness, the case of the square lattice for which the weights of the \((i, j)\)th vertex are given in terms of the variable

\[
\tilde{u}_{ij} = v_i - u_j,
\]

where \( v_i \) and \( u_j \) are the associated line variables (cf. fig. 1). The corresponding Potts model then has interactions \( K_{ij} \) given by

\[
e^{K_{ij}} - 1 = \frac{\tilde{A}_{ij}}{\tilde{B}_{ij}},
\]

where

\[
\{\tilde{A}_{ij}, \tilde{B}_{ij}\} = \begin{cases} \{A_{ij}, B_{ij}\}, & i + j = \text{even}, \\ \{B_{ij}, A_{ij}\}, & i + j = \text{odd}, \end{cases}
\]

\[
A_{ij} = \sinh \tilde{u}_{ij}/\sinh \eta,
\]

\[
B_{ij} = \sinh(\eta - \tilde{u}_{ij})/\sinh \eta.
\]

The partition function is given by

\[
Z_{\text{Potts}}(Q) = Q^{M/2}Z_{\text{NIS}}(Q^{1/2}) \prod_{i,j} \tilde{B}_{ij},
\]

where \( M \) is the number of Potts spins.

For a Potts model with anisotropic interactions \( K_1 \) and \( K_2 \) (depending on whether \( i + j \) is even or odd, corresponding to "horizontal" or "vertical"
interactions in the diagonally oriented Potts model), relation (50) implies the solubility condition

$$(e^{K_1} - 1)(e^{K_2} - 1) = Q,$$

which is precisely the critical point. More generally, around an elementary square whose corners are at sites $(i, j), (i + 1, j), (i, j + 1), (i + 1, j + 1)$, we have the consistency condition

$$\tilde{u}_{ij} + \tilde{u}_{i+1,j+1} = \tilde{u}_{i,j+1} + \tilde{u}_{i+1,j}.$$  \hspace{1cm} (53)

The conditions (53) then specify a subspace in the parameter space $K_{ij}$ in which the Potts model is solved.

One example is the checkerboard Potts lattice for which there are four interaction parameters $K_{11}, K_{12}, K_{21}$ and $K_{22}$. The condition (53), which then reads

$$\tilde{u}_{11} + \tilde{u}_{22} = \tilde{u}_{12} + \tilde{u}_{21},$$  \hspace{1cm} (54)

leads to the critical point of the checkerboard Potts model first conjectured by one of us\textsuperscript{28}). The solution (33) now leads to the following explicit expression for the critical partition function per site of this Potts model:

$$\kappa_{\text{critical}} = Q^{1/2} \left[ \prod_{i=1}^{2} \prod_{j=1}^{2} \frac{\kappa(\tilde{u}_{ij})}{B_{ij}} \right]^{1/2},$$  \hspace{1cm} (55)

where $\kappa(\tilde{u})$ is given by (47). This expression of the critical energy agrees with that obtained by Rammal and Maillard\textsuperscript{29})\textsuperscript{*} and by Baxter\textsuperscript{30}).

6.2. Graphical interpretation of vertex weights

The vertex weights (3) and (16) of the soluble NIS model can be given a simple graphical interpretation. Consider the case of a rhombic lattice whose lattice edges make angles $\alpha$ and $\beta = \pi - \alpha$. Define

$$x_\rho = \begin{cases} 1, & 1 \leq \rho \leq n, \\ e^{-\rho/2\pi}, & n < \rho \leq q, \end{cases}$$  \hspace{1cm} (56)

and let $\phi$ (or $\alpha$) be the angle between the two edges of the same color $\rho$ (or $\sigma$), then we find that we can rewrite vertex weights (3) and (16) as

\* Apart from a missing square power in (3.1) of ref. 29.
Thus, we may regard \( x_\rho \) as the per-radian weight associated to a corner of color \( \rho \). This graphical interpretation of the vertex weights may provide a key to a more direct proof of the equivalence of the solution of the \( n \neq q \) models with that of the separable NIS models.

Remark. Baxter and Forrester\(^{31} \) have considered a few small finite lattices which led them to speculate that the above-mentioned equivalence should hold for finite lattices of arbitrary size under suitable boundary conditions. In fact, such an analysis explains the structure of the model. The boundary condition can be chosen such that at boundary sites of valence two we assign a weight \( x_\rho^\phi \) with \( \phi = \alpha \) or \( \beta \) the angle closed in. Considering the case with one elementary square, (5) results. The next case with two elementary squares can be decomposed into one or two loops. The equivalence can only hold if for each loop the total angle is \( 2\pi \), explaining the opposite signs in (57). The explicit form (57) follows after considering cases with loops inside loops of conserved color, which implies that \( W_{\rho\rho} \) should be independent of the two sign choices in (57). The only additional freedom allowed is \( x_\rho = \pm 1 \) or \( e^{\frac{\pi}{2}\eta} \) leading to the complex cases discussed in ref. 3.

6.3. Oriented NIS model

The oriented NIS (ONIS) model introduced in ref. 5 is a generalization of the NIS model to include oriented edges as well. With a little effort one can generalize the solution (33) to the ONIS model.

The ONIS model, whose vertex weights and the associated vertex configurations have been shown in fig. 4 of ref. 5, is characterized by parameters \( q_1 \), the number of colors which do not admit orientation, \( q_2 \), the number of colors accompanied by arrows, and the \( q_2 \) fixed parameters \( z_\mu, \mu = 1, 2, \ldots, q_2 \), which give the additional weight factors per radian of turn of arrow of color \( \mu \) at a vertex. [The case \( q_2 = 0 \) is the separable NIS \( (n = q) \) model; the case \( q_1 = 0, q_2 = 1 \) is the ice-rule model.] For this ONIS model we have established the following: If we take

\[
A_i = \frac{\sinh u_i}{\sinh \eta}, \quad B_i = \frac{\sinh(\eta - \bar{u}_i)}{\sinh \eta},
\]

where

\[
W_{\rho\rho} = x_\rho^{\frac{\eta}{\eta - \bar{u}_i}} \frac{\sinh(\eta/\pi)}{\sinh \eta} + x_\rho^{\frac{\eta}{\eta - \bar{u}_i}} \frac{\sinh(\eta/\pi)}{\sinh \eta},
\]

\[
W_{\rho\sigma} = (x_\rho x_\sigma)^\phi \frac{\sinh(\eta(\pi - \phi)/\pi)}{\sinh \eta}, \quad \rho \neq \sigma.
\]
\[ 2 \cosh \eta = q_1 + \sum_{\mu=1}^{q_2} (z_{\mu}^{2w} + z_{\mu}^{-2w}) , \]  

(59)

Remark. One can further verify that there is a more general NIS model satisfying the star–triangle equation. In this general model again we have \( q_1 \) nonoriented colors and \( q_2 \) oriented colors with associated \( z_{\mu} \)'s. But now there exists an \( n \) with \( 0 \leq n \leq q_1 \) such that all weights without arrow are given by (3), (4), and (16) with \( q \) replaced by \( q_1 \) and \( \eta \) given by (5) with \( q \) replaced by the RHS of (59). The remaining weights are given in fig. 4 of ref. 5, except the ones with one oriented color \( \mu \) and one nonoriented color \( \rho \) with \( n < \rho \leq q_1 \). In this case there is an additional factor as in the \( a_{\rho} \) and \( b_{\rho} \) of (16), so all weights with two or four arrows are separable. For all these models we find the same inversion relation and partition function.

Finally, we note that there are many more solvable oriented NIS models of different nature. Some have already been found. One is the hard hexagon model\(^9\,^{32} \), which is an oriented NIS model with \( q_1 = q_2 = 1 \). Here the particles and holes are on the faces of the lattice and the allowed configurations map one-to-one by letting the oriented bonds encircle the particles clockwise, leading to only seven possible vertex configurations. More generally, the restricted SOS model\(^{33,\,34} \) with \( r - 1 \) different height values is an oriented NIS model with \( q_1 = 0 \) and \( q_2 = r - 2 \). Here the color of a bond is the lowest of the two heights of the faces separated by the bond. The arrow points in the direction corresponding to going around the face of lower height clockwise.

6.4. Comments and speculations on the inversion relation

In the present paper we have only solved the inversion relation (31) for the case of hyperbolic functions with \( \eta \) real, corresponding to Potts models with \( Q \geq 4 \) states. However, this case is known to be fairly representiative. Cases with hyperbolic or trigonometric functions follow as limiting cases from those with elliptic functions, by letting an appropriate period of the elliptic function tend to infinity. But this elliptic function generates an infinite product in the inversion relation and, consequently, the resulting partition function is an infinite product of factors, obtained in the same way as in the case with hyperbolic functions.

The inversion relation (31) follows quite generally from the star–triangle equation using (20). And its solution uses the symmetry (34). But there are
solutions to the star–triangle equation which do not satisfy (20) or (34). However, usually we can derive two inversion relations. For the hard-hexagon model\(^3\), for example, (20) still holds on the allowed states. There are now also ways\(^3,35\) to overcome the lack of the symmetry (34). Quite generally, one expects that the partition function per site is described by a pair of inversion relations of the form

\[
\kappa(u)\kappa(-u) = C \prod_{l=1}^{L} H(\lambda_l - u)H(\lambda_l + u),
\]

\[
\kappa(\zeta + u)\kappa(\zeta - u) = C' \prod_{l=1}^{L} H(\mu_l - u)H(\mu_l + u),
\]

where \(C\) and \(C'\) are constants, \(H(u)\) is a theta function (in the elliptic function case). This is certainly the case for the solutions of the star–triangle equations found so far, see for example, refs. 3, 9, 33–41. In cases where (34) holds (61) is a consequence of (60). It is very tempting to speculate that these two inversion relations follow generally. Then, one can solve them addressing one factor at a time. This will then lead to a solution \(\kappa(u)\) which is a product of solutions of the coupled equations

\[
\kappa(u)\kappa(-u) = cH(\lambda - u)H(\lambda + u),
\]

\[
\kappa(\zeta + u)\kappa(\zeta - u) = c'H(\mu - u)H(\mu + u).
\]

Since we also need to satisfy certain commensurability conditions among the different periods in the problem, this allows very few standard forms for the free energy. An explicit demonstration of this follows from the recent generalization of the hard-hexagon model by Baxter and Andrews\(^4\) and by Kuniba, Akutsu, and Wadati\(^4\). The resulting free energy agrees completely with the one found by Fateev\(^3\) in a 3-state generalization of the eight-vertex model\(^12,13,36\). However, there is no reason to expect that all other physical quantities agree. It looks more like the situation of representations of groups. Some quantities will depend on the representation, but the free energy does not. It is hoped that solvable models can be classified along such lines, and that it can be shown that for models that are representations of each other the free energies equal with suitable boundary conditions for lattices of arbitrary size.

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