An infinite-range bond percolation

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A bond percolation on a lattice in which all pairs of vertices are connected is considered. The percolation problem is treated by carrying out the Kasteleyn-Fortuin formulation of taking the q = 1 limit of a related q-component Potts model, the latter is exactly solved. For a lattice of N sites, an average of pN occupied bonds and N large, it is found that the system is percolating for p > 1/2. Closed-form expression is also obtained for the moment-generating function of the cluster size. The analysis yields the mean-field exponents
\( \beta = \gamma = \gamma' = 1 \).

PACS numbers: 05.50. + q, 05.70.Jk

It is customary to regard the Bethe-lattice solution of the percolation problem as its mean-field approximation [1]. While the Bethe-lattice consideration has led to yield the mean-field exponents, it has not been very helpful in providing an useful picture in understanding the mean-field nature of the percolation transition. In this paper we consider a model of a bond percolation, which is an extension of the usual mean-field spin models. By solving this percolation problem exactly, we have a picture of a mean-field percolation which is on the same footing as those of the spin systems.

It is well-known that a meaningful mean-field model for spin systems is one in which all spins interact with equal strength of the order of 1/N, N being the total number of spins [2]. In the same spirit, we therefore picture a mean-field model for bond percolation as a percolation process in which all pairs of sites are connected by bonds with an independent probability, also of the order of 1/N. Specifically, consider a system of N sites subject to a percolation process in which every pair of sites can be connected by a bond with a probability 2p/N; then on the average there are pN bonds. Thus this is a percolation with long-range interactions. It is known that such a percolation describes the problem of random graphs [3], and can be treated using a purely probabilistic approach [4]. Here we utilize the method of statistical physics by first considering a related Potts model [5]. The Potts model is then solved to give results on the percolation problem.

Consider a system of N q-state Potts spins whose Hamiltonian reads
\[ H = \sum_{<i,j>^+} \delta(q_i,q_j) + \frac{1}{2} \sum_{<i,j>^-} \delta(q_i,q_{\bar{j}}) \] (1)
where we have taken \( kT = 1 \), \( q_i = 1,2,\ldots,q \) denotes the spin states at the ith site, \( i = 1,2,\ldots,N \) and \( \delta \) is the Kronecker delta function. The summations \( <i,j>^+ \) and \( <i,j>^- \) are taken between all pairs of spins.

It is well-known [5,6] that the q = 1 limit of a Potts model generates a percolation. Particularly the Potts Hamiltonian (1) generates the long-range percolation under consideration with the following cluster-size generating function [6]:
\[ A(q;K,L) = \max \left\{ \mathbb{E}(x_1) - \ln x_1 \right\} \] (4)
\[ \{x_1\} \]

where
\[ \mathbb{E}(x_1) = \frac{K}{2} \left( \ln x_1^2 + \ln x_1 \right) \] (5)
is the "energy" of the system computed from (1) and subject to \( \{x_1\} \).

It is now straightforward to compute \( A(q;K,L) \) from (4) and (5). The result yields [7]
\[ A(q;K,L) = 1 - s - p(1-s)^2 \] (6)
where \( s_0 \) is determined from
\[ 2ps_0 + L + \ln(1-s_0) = 0 \] (7)

Quantities of interest in percolation can now be computed by taking the respective derivatives of \( A(q;K,L) \). Thus, we find the percolation probability, \( P(p) \), and the mean cluster size, \( S(p) \), to be given by [6]
\[ P(p) = 1 + G'(0+) = s_0 \] (8)
\[ S(p) = G'(0+) = (1-s_0)/(1-2p+2ps_0) \] (9)

Now, for \( p < 1/2 \), (7) with \( L = 0 \) has only one solution \( s = 0 \), hence \( P(p) = 0 \) identically. For \( p > 1/2 \), however, a second solution \( s > 0 \) arises and there is a nonzero percolation probability \( P(p) > s_0 \). The average cluster size is given by (9). Near the threshold we find
\[ P(p) \approx p - 1/2 \] (10)
\[ S(p) \approx (p - 1/2)^{-1} \] (11)

This leads to the mean-field exponents \( \beta = \gamma = \gamma' = 1 \), in agreement with the finding of [1]. We also note that the critical value of \( p = 1/2 \) coincides with the finding of the purely probabilistic approach [4].

REFERENCES

a) Supported in part by the National Science Foundation