LETTER TO THE EDITOR

Number of spanning trees on a lattice†

F Y Wu
Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA

Received 28 March 1977

Abstract. The number of spanning trees on a large lattice is evaluated exactly for the square, triangular and honeycomb lattices.

A spanning tree of a lattice \( \mathcal{L} \) is a graph drawn on \( \mathcal{L} \) which connects all lattice sites and contains no polygons. For a regular lattice of \( N \) sites, the number of spanning trees on \( \mathcal{L} \), \( T_N \), behaves as \( e^{zN} \) for large \( N \). We report here the exact values of \( z \) for the square (SQ), triangular (TR) and honeycomb (HC) lattices.

More specifically let

\[
 z = \lim_{N \to \infty} N^{-1} \ln T_N. \tag{1}
\]

We find

\[
 z_{SQ} = \frac{4}{\pi} (1 - 3^{-2} + 5^{-2} - 7^{-2} + \ldots) = 1.1662436 \ldots
\]

\[
 z_{TR} = \frac{3\sqrt{3}}{\pi} (1 - 5^{-2} + 7^{-2} - 11^{-2} + 13^{-2} - \ldots) = 1.61532968 \ldots \tag{2}
\]

\[
 z_{HC} = \frac{1}{2} z_{TR} = 0.8076648 \ldots.
\]

It was first pointed out by Fortuin and Kasteleyn (1972) that \( T_N \) is expressible in terms of the partition function of a lattice statistical model. For our purpose, it suffices to consider the following graph generating function on \( \mathcal{L} \):

\[
 Z_N(q, v) = \sum_G q^n v^e. \tag{3}
\]

Here the summation extends over all graphs \( G \) on \( \mathcal{L} \); \( n \) and \( e \) are, respectively, the numbers of clusters and edges in \( G \). \( Z_N \) is proportional to the cluster generating function of Fortuin and Kasteleyn (1972), and coincides with the partition function of a \( q \)-component Potts model for integral \( q \) (Baxter 1973). Now let \( v = q^\alpha \) and consider the \( q \to 0 \) limit of the function

\[
 Z_N(q, q^\alpha) = q^{\alpha N} \sum_G q^{n\alpha + (1-\alpha)e} \tag{4}
\]

† Work supported in part by NSF Grant No. DMR 76-20643.
where we have used the Euler relation $N + c = n + e$ to eliminate the parameter $e$ in favour of $c$, the number of independent circuits in $G$. For $\alpha = 1$ the leading terms in (4) in the $q \to 0$ limit are the tree graphs ($c = 0$). Consequently, $Z_N(q, q)$ generates forests of trees on $\mathcal{L}$ (Stephen 1976). For $0 < \alpha < 1$ the leading terms are the spanning trees ($c = 0, n = 1$). Thus we have the exact relation valid for any finite lattice

$$T_N = \lim_{q \to 0} q^{*(1-N)}^{-1} Z_N(q, q^{*}), \quad 0 < \alpha < 1.$$  \hspace{1cm} (5)

Equation (5) reduces to (7.13) of Fortuin and Kasteleyn (1972) upon taking $\alpha = \frac{1}{2}$.

For planar $\mathcal{L}$ the generating function (3) is related to the partition function of an ice-type problem on a related medial lattice $\mathcal{L}'$ (Baxter et al. 1976). The choice of $\alpha = \frac{1}{2}$ in (5) is especially convenient, for the resulting ice-type model is well defined and soluble in the $q \to 0$ limit. Combining (5) with (14) of Baxter et al. (1976), we obtain (with $\alpha = \frac{1}{2}$) from (1)

$$z = \lim_{N \to \infty} N^{-1} \ln Z'$$  \hspace{1cm} (6)

where $Z'$ is the partition function of an ice-type model defined on the medial lattice $\mathcal{L}'$. If $\mathcal{L}$ is a square lattice, then $\mathcal{L}'$ is also a square lattice but having $2N$ sites. The vertex weights of the ice-type model on $\mathcal{L}'$ are (cf figure 5 of Baxter et al. 1976)

$$\omega_1, \ldots, \omega_6 = 1, 1, 1, 1, \sqrt{2}, \sqrt{2}.$$  \hspace{1cm} (7)

If $\mathcal{L}$ is a triangular (honeycomb) lattice, then $\mathcal{L}'$ is a Kagomé lattice of $3N$ ($3N/2$) sites with the following vertex weights:

$$\omega_1, \ldots, \omega_6 = 1, 1, 1, 1, e^{-i\pi/6} + e^{i\pi/3}, e^{i\pi/6} + e^{-i\pi/3}.$$  \hspace{1cm} (8)

In either case, it is readily verified that the weights satisfy the free-fermion conditions (Fan and Wu 1970)

$$\omega_{12} + \omega_{34} = \omega_5 \omega_6$$  \hspace{1cm} (9)

so that the right-hand side of (6) can be evaluated by computing a Pfaffian. In the case of square lattice, the free-fermion solution of $Z'$ was first evaluated by Wu and reported in Lieb (1967). The numerical value (2) for $z_{SQ}$ now follows from (20) of Lieb (1967) and the fact that $\mathcal{L}'$ contains $2N$ sites. In the case of Kagomé lattice, the free-fermion solution of $Z'$ has been obtained by Lin (1975)†. In the notation shown in figure 2 of Lin (1975), we may rewrite (8) as

$$\omega_i = \omega'_i = \omega''_i, \quad i = 1, 2, 3, 4$$
$$\omega_5 = \omega'_5 = \omega''_5 = e^{-i\pi/6} + e^{i\pi/3}$$
$$\omega_6 = \omega'_6 = \omega''_6 = e^{i\pi/6} + e^{-i\pi/3}$$  \hspace{1cm} (10)

Equation (11) of Lin (1975)† now leads to

$$z_{TR} = 2z_{HC} = \frac{1}{4\pi} \int_{0}^{2\pi} d\theta \int_{0}^{2\pi} d\phi \ln[6 - 2 \cos \theta - 2 \cos \phi - 2 \cos(\theta + \phi)].$$  \hspace{1cm} (11)

This reduces to (2) upon carrying out the integrations.

† The free energy given by Lin (1975) contains an error. The right-hand side of his equation (11) (and all other expressions for $\phi$) should be multiplied by a factor $\frac{1}{3}$.
References

Fortuin C M and Kasteleyn P W 1972 Physica 57 536-64
Stephen M 1976 Phys. Lett. 56A 149-50