Seven coefficients in the high temperature series expansions for the zero-field susceptibility and the specific heat are derived for the planar classical Heisenberg model with biquadratic interactions. The critical temperatures and the susceptibility exponents are determined for cubic lattices.

In recent years there has been considerable interest in the critical properties of spin systems with biquadratic interactions. With the presence of biquadratic interactions, the critical temperature and the critical indices of the system change\(^1\). When the biquadratic interactions are strong enough the phase transition may become first order and phase transitions associated with the quadrupole moments may occur\(^2\).

In this paper we consider the planar classical Heisenberg model with biquadratic interactions. The Hamiltonian is given by

\[
\mathcal{H} = - \sum_{\langle ij \rangle} \left[ 2J_1 \mathbf{S}_i \cdot \mathbf{S}_j + 2J_2 (\mathbf{S}_i \cdot \mathbf{S}_j)^2 \right] - m \mathbf{H} \cdot \sum_i \mathbf{S}_i,
\]

where \(\mathbf{S}_i\) is a two-dimensional unit vector at the lattice site labelled \(i\), \(m\) is the magnetic moment per spin, \(J_1\) and \(J_2\) are respectively the bilinear and biquadratic interaction constants, and the first summation is taken over all pairs of nearest-neighbor sites. The direction of the external magnetic field \(\mathbf{H}\) is taken parallel to the planes containing the spins.

For \(J_2 = 0\) the system has been studied by Bowers and Joyce\(^3\). They used the method of exact high temperature series expansions to determine the critical temperature and critical indices. The high temperature series for the susceptibility \(\chi_0\) and the specific heat \(C_0\) at zero field are of the forms

\[
\chi_0 = \frac{(Nm^2/2kT)}{1 + \sum_{n=1}^{\infty} a_n (J_1/kT)^n},
\]

\[
C_0 = \frac{(Nm^2/2kT^2)}{1 + \sum_{n=1}^{\infty} a_n (J_1/kT)^n},
\]

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\[ C_0 = Nk \sum_{n=2}^{\infty} b_n (J_1 / kT)^n, \]

where the coefficients \( a_n \) and \( b_n \) are functions of \( \eta (= J_2 / J_1) \). For \( J_2 = 0 \) coefficients through \( a_8 \) and \( b_8 \) for the cubic lattices were obtained by Bowers and Joyce. For general values of \( J_2 \) we have derived the coefficients \( a_1 - a_7 \) and \( b_2 - b_8 \) for the cubic lattices. These coefficients are obtained by the technique developed by Joyce to derive the high temperature series for the classical Heisenberg model. Our results are as follows: For the s.c. lattice

\[
\begin{align*}
a_1(\eta) &= 6, \\
a_2(\eta) &= 30 + 3\eta, \\
a_3(\eta) &= 147 + 30\eta, \\
a_4(\eta) &= 696 + 233\eta + 13.5\eta^2 + 2.625\eta^3, \\
a_5(\eta) &= 3275 + 1505\eta + 186.125\eta^2 + 26.25\eta^3, \\
a_6(\eta) &= 15171.5 + 9103.125\eta + 1727.25\eta^2 + 267.625\eta^3 + 25.125\eta^4 + 6.0625\eta^6, \\
a_7(\eta) &= 70009.125 + 51461.25\eta + 13575.975\eta^2 + 2443.5\eta^3 + 363.22916\eta^4 + 60.625\eta^5, \\
\end{align*}
\]

and

\[
\begin{align*}
b_2(\eta) &= 6 + 1.5\eta^2, \\
b_3(\eta) &= 9\eta, \\
b_4(\eta) &= 63 + 3.9375\eta^4, \\
b_5(\eta) &= 220\eta + 26.25\eta^3, \\
b_6(\eta) &= 970 + 327.1875\eta^2 + 15.1236975\eta^6, \\
b_7(\eta) &= 4963.875\eta + 404.25\eta^3 + 127.05\eta^5, \\
b_8(\eta) &= 13395.375 + 12822.6\eta^2 + 1215.9583\eta^4 + 52.339826\eta^8. \\
\end{align*}
\]

For the b.c.c. lattice

\[
\begin{align*}
a_1(\eta) &= 8, \\
a_2(\eta) &= 56 + 4\eta, \\
a_3(\eta) &= 388 + 56\eta, \\
a_4(\eta) &= 2592 + 630.667\eta + 38\eta^2 + 11.5\eta^3, \\
a_5(\eta) &= 17230.667 + 5817.333\eta + 688.167\eta^2 + 161\eta^3, \\
a_6(\eta) &= 112843.333 + 50241.5\eta + 8733\eta^2 + 2084.167\eta^3 + 173.5\eta^4 + 46.5833\eta^5, \\
a_7(\eta) &= 736900.167 + 404087.667\eta + 94877.30\eta^2 + 23350\eta^3 + 3364.4722\eta^4 + 652.167\eta^5, \\
\end{align*}
\]

and

\[
\begin{align*}
b_2(\eta) &= 8 + 2\eta^2, \\
b_3(\eta) &= 12\eta, \\
b_4(\eta) &= 276 + 17.25\eta^4, \\
\end{align*}
\]
HEISENBERG MODEL WITH Biquadratic INTERACTIONS

\[ b_5(\eta) = 933.333\eta + 115\eta^3, \]
\[ b_6(\eta) = 7453.333 + 1876.25\eta^2 + 116.4149\eta^6, \]
\[ b_7(\eta) = 41450.5\eta + 5635\eta^3 + 977.9\eta^5, \]
\[ b_8(\eta) = 218919.167 + 130179.46\eta^2 + 14909.611\eta^4 + 855.17176\eta^8. \]

For the f.c.c. lattice

\[ a_1(\eta) = 12, \]
\[ a_2(\eta) = 132 + 6\eta, \]
\[ a_3(\eta) = 1398 + 156\eta + 12\eta^2, \]
\[ a_4(\eta) = 14496 + 2734\eta + 363\eta^2 + 32.25\eta^3, \]
\[ a_5(\eta) = 148294 + 40378\eta + 7451.25\eta^2 + 1036.5\eta^3 + 97.5\eta^4, \]
\[ a_6(\eta) = 150306 + 541487.25\eta + 127438.5\eta^2 + 22974.75\eta^3 + 3243.75\eta^4 + 316.25\eta^5, \]
\[ a_7(\eta) = 15132379.25 + 6827181.138\eta + 1953405.448\eta^2 + 426479\eta^3 + 75532.833\eta^4 + 10822.75\eta^5 + 1091.125\eta^6. \]

\[ b_2(\eta) = 12 + 3\eta^2, \]
\[ b_3(\eta) = 96 + 18\eta + 12\eta^3, \]
\[ b_4(\eta) = 774 + 288\eta + 72\eta^2 + 48.375\eta^4, \]
\[ b_5(\eta) = 6240 + 3440\eta + 960\eta^2 + 322.5\eta^3 + 195\eta^5, \]
\[ b_6(\eta) = 50600 + 37200\eta + 13111.875\eta^2 + 3870\eta^3 + 1462.5\eta^4 + 790.5599\eta^6, \]
\[ b_7(\eta) = 418992 + 379086.75\eta + 164346\eta^2 + 53140.5\eta^3 + 16380\eta^4 + 6640.725\eta^5 + 3273.375\eta^7, \]
\[ b_8(\eta) = 3543499.75 + 3752877.34\eta + 1932501.2\eta^2 + 696976\eta^3 + 224183.167\eta^4 + 70812\eta^5 + 30551.5\eta^6 + 13841.824\eta^8. \]

\[ \text{TABLE I} \]

Critical temperatures \( kT_c/J_1 \) and critical indices \( \gamma \) for cubic lattices

<table>
<thead>
<tr>
<th>( J_2/J_1 )</th>
<th>( \text{f.c.c.} )</th>
<th>( \text{b.c.c.} )</th>
<th>( \text{s.c.} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( kT_c/J_1 )</td>
<td>( \gamma )</td>
<td>( kT_c/J_1 )</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>0</td>
<td>9.65</td>
<td>1.31</td>
<td>6.25</td>
</tr>
<tr>
<td>0.1</td>
<td>9.76</td>
<td>1.29</td>
<td>6.34</td>
</tr>
<tr>
<td>0.2</td>
<td>9.87</td>
<td>1.27</td>
<td>6.41</td>
</tr>
<tr>
<td>0.3</td>
<td>10.00</td>
<td>1.25</td>
<td>6.49</td>
</tr>
<tr>
<td>0.4</td>
<td>10.15</td>
<td>1.23</td>
<td>6.58</td>
</tr>
<tr>
<td>0.5</td>
<td>10.24</td>
<td>1.21</td>
<td>6.64</td>
</tr>
<tr>
<td>0.6</td>
<td>10.37</td>
<td>1.19</td>
<td>6.72</td>
</tr>
<tr>
<td>0.7</td>
<td>10.41</td>
<td>1.18</td>
<td>6.81</td>
</tr>
<tr>
<td>0.8</td>
<td>10.63</td>
<td>1.16</td>
<td>6.90</td>
</tr>
<tr>
<td>0.9</td>
<td>10.78</td>
<td>1.14</td>
<td>7.05</td>
</tr>
<tr>
<td>1.0</td>
<td>10.92</td>
<td>1.13</td>
<td>7.14</td>
</tr>
</tbody>
</table>

\( \dagger \) The uncertainties in the estimates of \( kT_c/J_1 \) and \( \gamma \) are within \( \pm 0.01 \) and \( \pm 0.02 \), respectively.
The above series are analysed by the ratio and Padé methods\textsuperscript{5}). The critical temperatures $kT_c/J_1$ and the susceptibility exponents $\gamma$ for several values of $\eta \leq 1$ are shown in table I. For $\eta > 1$ the mean-field theory predicts a first order phase transition, and estimates of $T_c$ and $\gamma$ from the high temperature series may be incorrect. We see from table I that when the strength of biquadratic interactions increases, the critical temperature increases, while the susceptibility exponent decreases. The critical index $\gamma$ depends strongly on the relative strength of the interactions. This property is similar to that found for the Ising model with quadrupolar crystal field\textsuperscript{6}). The uncertainty in the estimate of the specific heat exponent $\alpha$ is very large. However, it is evidence that $\alpha$ is an increasing function of $\eta$ as $\eta$ varies from 0 to 1.

References