Staggered eight-vertex model

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An eight-vertex model with staggered (site-dependent) vertex weights is considered. The model is an extension of the usual one with translationally invariant weights and contains sixteen independent vertex weights. From its Ising representation it is seen that there are actually only eleven independent parameters. After discussing some general symmetry properties of this model, we consider in detail the soluble case of a free-fermion model. We find that the staggered free-fermion model may exhibit up to three phase transitions. Generally the specific heat has logarithmic singularities, except in some special cases it has an exponent $\alpha = 1/2$ and the system is frozen below a unique transition point. Conditions for these special cases are given.

I. INTRODUCTION

In the consideration of vertex models in lattice statistics, one usually deals with models whose vertex weights are translationally invariant. In a recent paper, hereafter referred to as I, we have pointed out the importance of models with staggered, or site-dependent, weights. We have also studied the most general Pfaffian solution of the staggered six-vertex model. In this paper, we return to consider the staggered eight-vertex model.

We refer to I for the definition of a staggered model. For an eight-vertex model, the allowed vertex configurations can be conveniently described by the bond graphs shown in Fig. 1.\* Let the vertex weights be

$$\{\omega\} = \{\omega_1, \omega_2, \ldots, \omega_8\} \quad \text{on sublattice A},$$
$$\{\omega'\} = \{\omega'_1, \omega'_2, \ldots, \omega'_8\} \quad \text{on sublattice B}. \quad (1)$$

Then, as in I, our goal is to compute the “free energy”

$$\psi = \lim_{N \to \infty} \frac{1}{N} \ln Z, \quad (2)$$

where $Z$ is the partition-generating function defined by Eq. (1) and $N$ is the number of sites. In the ferroelectric language, the vertex weights are the Boltzmann factors

$$\omega_i = \exp(-\beta e_i), \quad \omega'_i = \exp(-\beta e'_i), \quad (3)$$

where $\beta = 1/kT$ and $e_i, e'_i$ are the vertex energies. This completes the definition of the staggered eight-vertex model.

When $\omega_1 = \omega_2 = \omega_3 = 0$, this model reduces to the ice-rule case considered in I. It is also known\* that for

$$e_1 = e_2 = e'_1 = e'_2, \quad e_3 = e_4 = e'_3 = e'_4, \quad (4)$$
$$e_5 = e_6 = e'_5 = e'_6, \quad e_7 = e_8 = e'_7 = e'_8,$$

the model is equivalent to the Ashkin-Teller model,\* which generally possesses two phase transitions.\*\* In other cases, as we shall see, the model may also exhibit three transitions. Thus the staggered eight-vertex model is very general and may find applications to systems with multiple phase transitions.

The outline of this paper is as follows: Some symmetry properties of the model are discussed in Sec. II. In Sec. III we consider the Ising representation of the staggered eight-vertex model, thereby showing the existence of only eleven independent parameters. In Sec. IV we study the staggered free-fermion model and summarize its thermodynamic properties. Details of analyses are found in Sec. V. A striking feature of our result is the existence of multiple phase transitions.

II. SYMMETRY RELATIONS

The partition function $Z$ possesses a number of symmetry relations which can be obtained by quite general considerations. Obviously, $Z$ is invariant if $\omega_i$ and $\omega'_i$, $i = 1, 2, \ldots, 8$ are interchanged. This is written as

$$Z = Z(12\ldots 8; 1'2'\ldots 8')$$
$$= Z(1'2'\ldots 8'; 12\ldots 8), \quad (5)$$

where $i$ and $i'$ denote, respectively, $\omega_i$ and $\omega'_i$. Other symmetries can be obtained along the line of considerations for an eight-vertex model with uniform weights, the uniform model.\*\* Interchanging the bonds and “holes” in the horizontal or the vertical directions, e.g., we obtain

$$Z = Z(43217856; 4'3'2'1'7'8'5'6'),$$
$$= Z(34128765; 3'4'1'2'8'7'6'5'). \quad (6)$$

A 90° clockwise rotation of the lattice leads to

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\[ Z = Z(12438756; 1'2'3'4'5'6'). \]  

(7)

Interchanging the bonds and holes along the zigzag paths shown in Fig. 2 of Ref. 6 gives the symmetry

\[ Z = Z(56782134; 5'6'7'8'1'2'3'). \]  

(8)

The last relation reveals that the pairs of vertex weights

\[ \{ \omega_6, \omega_6' \}, \{ \omega_5, \omega_5' \}, \{ \omega_4, \omega_4' \} \]  

(9a)

play the same role as

\[ \{ \omega_1, \omega_1' \}, \{ \omega_2, \omega_2' \}, \{ \omega_3, \omega_3' \}, \{ \omega_4, \omega_4' \}, \]  

(9b)

a property also reflected in Eq. (13) below. Finally, there is the weak-graph symmetry\(^5\) which is local property of a lattice and is therefore valid even if the weights are site dependent. If the conjugate vertex pairs\(^5\) have equal weights

\[ \omega_1 = \omega_2 = a_1, \quad \omega_2' = \omega_6 = a_2, \]

\[ \omega_2 = \omega_4 = b_1, \quad \omega_6' = \omega_6 = b_2, \]

\[ \omega_5 = \omega_5' = c_1, \quad \omega_5 = \omega_6' = c_2, \]

\[ \omega_7 = \omega_8 = d_1, \quad \omega_7 = \omega_8 = d_2, \]

(10)

this symmetry leads to, in obvious notations,

\[ Z(a_1 b_2 c_1 d_1; a_2 b_2 c_2 d_2) = Z(a_1 b_1 c_2 d_1; a_2 b_2 c_2 d_2). \]  

(11)

Here, for \( i = 1, 2, \)

\[ a_i = \frac{1}{2}(a_1 + b_1 + c_i + d_i), \quad b_i = \frac{1}{2}(a_1 + b_1 - c_i - d_i), \]

\[ c_i = \frac{1}{2}(a_1 + b_1 + c_i - d_i), \quad d_i = \frac{1}{2}(a_1 + b_1 - c_i + d_i). \]  

(12)

The case \( a_1 = a_2, \quad b_1 = b_2, \quad c_1 = c_2, \quad d_1 = d_2 \) is equivalent to the Ashkin-Teller model\(^5\).

III. EQUIVALENCE WITH AN ISING MODEL

Although there are 16 vertex weights in the definition (1), it turns out that the vertex weights enter the free energy \( \psi \) through the combinations of

\[ v_1 = \omega_1 \omega_1', \quad v_2 = \omega_2 \omega_2', \quad v_3 = \omega_3 \omega_3', \quad v_4 = \omega_4 \omega_4', \]

\[ v_5 = \omega_5 \omega_5', \quad v_6 = \omega_6 \omega_6', \quad v_7 = \omega_7 \omega_7', \quad v_8 = \omega_8 \omega_8', \]

\[ n = \omega_1 \omega_2 \omega_3 \omega_4; \quad v = \omega_1 \omega_4 \omega_5 \omega_6; \quad w = \omega_1 \omega_4 \omega_7 \omega_8. \]  

(13)

In other words, there are only eleven independent parameters in a staggered eight-vertex model.

\[ \begin{array}{cccccccc}
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
A & \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 & \omega_6 & \omega_7 & \omega_8 \\
B & \omega_1' & \omega_2' & \omega_3' & \omega_4' & \omega_5' & \omega_6' & \omega_7' & \omega_8' \\
\end{array} \]

FIG. 1. The eight-vertex configurations and the associated weights.

To see this result, we consider the Ising representation of an eight-vertex model\(^1,9\). In a staggered model, the Ising interactions are also staggered. Let the four spins surrounding a site be \( \sigma_1, \sigma_2, \sigma_3, \) and \( \sigma_4 \), as shown in Fig. 2. They interact with two-spin interactions \(-J_1^{ij}\) and four-spin interaction \(-J_{1234}^{\alpha}\), where the superscript \( \alpha = A \) or \( B \) refers to the sublattices. Then, analogous to Eq. (29) of Ref. 1,\(^12\) the vertex energies can be written as

\[ e_\alpha(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = -J_1^{\alpha} - \sum_{i<j} J_1^{ij} \sigma_i \sigma_j - J_{1234}^{\alpha} \sigma_1 \sigma_2 \sigma_3 \sigma_4, \quad \alpha = A, B. \]  

(14)

Here, we have adopted the convention (cf. Fig. 1) of writing \( e_1 = e^A(1, 1, 1, 1) = e^A(-1, -1, -1, -1), \)

\[ e_1' = e^B(1, 1, 1, 1) = e^B(-1, -1, -1, -1), \]

\( e_2 = e^A(1, -1, -1, -1), \) etc. Since Eq. (14) is invariant under \( \sigma_i \rightarrow -\sigma_i \), it represents sixteen independent equations. They can be solved to yield

\[ J_1^A = -\frac{1}{16} \sum_{1\leq i<j\leq 4} \sigma_i \sigma_j e^A, \]

\[ J_1^B = -\frac{1}{16} \sum_{1\leq i<j\leq 4} \sigma_i \sigma_j e^B, \]

\[ J_{1234}^A = -\frac{1}{16} \sum_{1\leq i<j<k<l\leq 4} \sigma_i \sigma_j \sigma_k \sigma_l e^A, \]

\[ J_{1234}^B = -\frac{1}{16} \sum_{1\leq i<j<k<l\leq 4} \sigma_i \sigma_j \sigma_k \sigma_l e^B. \]  

(15)

The resulting Ising lattice shown in Fig. 3 has eleven interactions:

\[ J_{12}^A, \quad J_{12}^B, \quad J_{1234}^{\alpha}, \quad (\alpha = A, B) \]

\[ J_0 = J_1^A + J_1^B, \]

\[ J_1 = J_{12}^A + J_{12}^B, \quad J_2 = J_{12}^A + J_{12}^B, \]

\[ J_3 = J_{1234}^A + J_{1234}^B. \]  

(16)

Writing out Eqs. (16) and using the variables defined in Eqs. (13), we find explicitly
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FIG. 3. Equivalent Ising lattice with staggered interactions. Four-spin interactions are not shown.

\[
\exp(-8\beta J_0) = \prod_{i=1}^{8} v_i,
\]

\[
\exp(8\beta J_1) = v_2v_3v_4v_5/v_1v_4v_5v_7,
\]

\[
\exp(8\beta J_2) = v_2v_3v_5v_7/v_1v_2v_4v_6,
\]

\[
\exp(8\beta J_3) = v_2v_4v_5v_7/v_1v_2v_3v_6,
\]

\[
\exp(8\beta J_4) = v_2v_4v_5v_6/v_1v_2v_3v_7,
\]

\[
\exp(8\beta J_5) = v_3v_4v_5v_6/v_1v_2v_3v_7,
\]

\[
\exp(8\beta J_6) = v_3v_4v_5v_7/v_1v_2v_3v_6,
\]

\[
\exp(8\beta J_7) = v_3v_4v_6v_7/v_1v_2v_3v_5,
\]

\[
\exp(8\beta J_8) = v_3v_5v_6v_7/v_1v_2v_4v_5.
\]

This completes the proof of our assertion (13).

IV. STAGGERED FREE-FERMION MODEL

In this section we study the most general Pfaffian solution of the staggered eight-vertex model. A vertex model is exactly soluble if a certain free-fermion condition is satisfied at each vertex. For the present model the condition reads

\[
\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8,
\]

\[
\omega_1'\omega_2' + \omega_3'\omega_4' = \omega_5'\omega_6' + \omega_7'\omega_8'.
\]

Under this condition, the partition function is equal to a Pfaffian which can be evaluated in a closed form. With the details outlined in Appendix A, the result is given by

\[
\psi = \frac{1}{16\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln F(\theta, \phi),
\]

with

\[
F(\theta, \phi) = F_0(\theta, \phi) - 4\Delta \sin^2 \phi - 4\Delta' \sin^2 \theta.
\]

We remark that, without the two terms involving \(\sin^2\phi\) and \(\sin^2\phi\) in Eq. (20), the solution is exactly of the form of that of a uniform eight-vertex model with weights \(\{\Omega_1, \ldots, \Omega_8\}\), which satisfy the free-fermion condition

\[
\Omega_1\Omega_2 + \Omega_3\Omega_4 = \Omega_5\Omega_6 + \Omega_7\Omega_8.
\]

Note that there are now only seven independent parameters in the solution (19).

It is instructive to demonstrate that the solution (19) indeed reduces to the previously known expressions. For \(\omega = \omega'\), the uniform model of Ref. 6, it is readily verified that \(F(\theta, \phi)\) factorizes into

\[
F(\theta, \phi) = [2a + 2b \cos \alpha + 2c \cos \beta + 2d \cos(\alpha - \beta)
\]

\[
+ 2e \cos(\alpha + \beta)][2a - 2b \cos \alpha - 2c \cos \beta
\]

\[
+ 2d \cos(\alpha + \beta) + 2e \cos(\alpha + \beta)],
\]

with

\[
\alpha = \frac{1}{2}(\theta + \phi), \quad \beta = \frac{1}{2}(\theta - \phi),
\]

\[
a = \frac{1}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2), \quad b = \omega_4\omega_1 - \omega_2\omega_3,
\]

\[
c = \omega_1\omega_4 - \omega_2\omega_3, \quad d = \omega_3\omega_6 - \omega_7\omega_8,
\]

\[
e = \omega_2\omega_6 - \omega_8\omega_7.
\]

The solution (19) now reduces to that of Ref. 6. For \(\theta = \phi = 0\), the staggered ice-rule model, \(F(\theta, \phi)\)
can be written as

\[
F(\theta, \phi) = |v_1 e^{i\theta} + v_2 e^{i\theta} + v_3 e^{i(\tau + \sigma)} + v_4 e^{i(\tau - \sigma)} + v_5 + v_6|^2. \tag{24}
\]

This is the result of I.

To analyze the analytic properties of Eq. (19), one could, as was done in I, carry out one of the two integrations in Eq. (19). Unfortunately the resulting expression is given in terms of the roots of a quartic equation, which does not appear to be very illuminating. We shall, therefore, be less ambitious and confine our considerations to the Boltzmann weights, Eq. (3), the physical model of a ferroelectric, and require the free-fermion condition (18) to hold at all temperatures.

To satisfy Eq. (18) at all temperatures, we may take, without loss of generality,

\[
\omega_1 \omega_2 = \omega_3 \omega_4, \quad \omega_1 \omega_5 = \omega_6 \omega_7.
\]

There are now two possibilities for the second condition in Eq. (18), to hold,

\[
(\text{I}) \quad \omega_7 \omega_8 = \omega_3 \omega_6; \quad \omega_7 \omega_9 = \omega_4 \omega_8 \quad \text{(26a)}
\]

\[
(\text{II}) \quad \omega_7 \omega_8 = \omega_3 \omega_6; \quad \omega_7 \omega_9 = \omega_4 \omega_8 \quad \text{(26b)}
\]

In both cases, we find the free energy \(\psi\) nonanalytic at \(T = T_c\), defined by

\[
\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 \geq 2\max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}. \tag{27}
\]

We also find that the singular part of \(\psi\) behaves as

\[
\psi_{\text{sing}} \sim i^2 \ln |t|, \quad t = (T - T_c)/T_c - 0, \tag{28}
\]

except for (a) \(\Omega_2 \Omega_3 \Omega_4 \Omega_5 = 0\) in case (I), and (b) \(\Omega_2 \Omega_3 = \Delta_1 = 0\) or \(\Omega_2 \Omega_4 = \Delta_2 = 0\) in case (ii), where

\[
\Delta_1 = \Omega_3 \Omega_4 \Omega_5 \Omega_6 - (\Omega_2 + \Omega_4)^2, \tag{29}
\]

\[
\Delta_2 = \Omega_3 \Omega_4 \Omega_5 \Omega_6 - (\Omega_1 + \Omega_4)^2.
\]

In these special cases, we find

\[
\psi = \text{frozen}, \quad T < T_c, \quad \psi_{\text{sing}} \sim i^{3/2}, \quad T > T_c. \tag{30}
\]

Detailed analysis leading to these results will be given in Sec. V. Note that the specific heat now has an Ising-type (logarithmic) singularity; except in cases (a) and (b), it has an exponent \(\alpha = \frac{1}{2}\). \(^{11}\)

One consequence of the critical condition (27) is that there may exist up to three phase transitions. To see this, let us take, without loss of generality,

\[
v_1 \geq v_2 \geq v_i, \quad i \neq 1, 2. \tag{31}
\]

If \(i = 2\), then Eq. (27) reads

\[
\Omega_1 = \Omega_2 + \Omega_3 + \Omega_4, \tag{32}
\]

which always has one solution. If \(i \neq 2\), say \(i = 3\), and \(\Omega_4\) is not a single Boltzmann factor, then in addition to Eq. (31), (27) can also be satisfied by

\[
\Omega_2 = \Omega_1 + \Omega_3 + \Omega_4. \tag{33}
\]

Since \(\Omega_2 \geq \Omega_1 + \Omega_3 + \Omega_4\) at \(T = 0\), \(\infty\) and both sides of Eq. (33) are monotonic and concave in \(T\), Eq. (33) has either 0 or 2 solutions. \(^{11}\) Therefore the staggered free-fermion model will, in general, possess up to three phase transitions. Note that Eq. (33) is not valid if \(\Omega_2\) is a single Boltzmann factor. Thus, there can be only one transition in the special cases (a) and (b). A well-known example of the staggered eight-vertex model which has three phase transitions is the Ising model on the Union Jack lattice. \(^{12}\) Let the first- and second-neighbor interactions be, respectively, \(-J\) and \(-J'\). Using Eq. (14), or more simply Eq. (29) of Ref. 1, we find the staggered vertex weights,

\[
\omega_1 = \omega = e^{b_J - a_J}, \quad \omega_2 = e^{b_J + a_J}, \quad \omega_3 = \omega_4 = e^{a_J}, \quad \omega_5 = \omega_6 = e^{b_J}. \tag{34}
\]

where \(K = J/kT, \quad K' = J'/kT\). It is readily verified that these weights satisfy the free-fermion condition (18). Thus Eqs. (32) and (33) read

\[
e^{-2k_J} \cosh 4K = e^{2k_J} + 2, \tag{35}
\]

\[
e^{2k_J} \cosh 4K = e^{2k_J'} + 2. \tag{36}
\]

The first equation has always one solution, while the second equation has two solutions for \(J'\) in the antiferromagnetic range \(-J < J' < 0, 0.9681 J\).

Before closing this section, we remark that there exists another soluble staggered eight-vertex model which exhibits two phase transitions. This is when the vertex weights are given by Eq. (10) and related by

\[
a_i/b_i = c_i d_i, \quad i = 1, 2. \tag{37}
\]

Although the vertex weights do not satisfy the free-fermion condition directly, the model is soluble because the Ising representation (15) decouples into a superposition of two nonequivalent nearest-neighbor Ising lattices. \(^{11-13}\) Consequently, the model has two distinct transitions.

V. ANALYTIC PROPERTIES OF $$\psi$$

Details that lead to the results quoted in Sec. IV are now given. It is convenient to consider the cases (I) and (II) or Eqs. (26a) and (26b) separately.

(I) Models satisfying the free-fermion conditions (25) and (26a)

The free energy is now given by

\[
\psi = \frac{1}{16\pi} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln F_0(\theta, \phi). \tag{38}
\]

As we have remarked earlier, Eq. (37) is now exactly of the form of the free energy of a uniform free-fermion model, \(^6\) the only difference being that
we do not have the simplification $D = 0$ or $E = 0$ used in Ref. 6. Nevertheless, the properties of $\psi$ can be similarly studied. Carry out the $\theta$ integration using, e.g., Eq. (21) of Ref. 6, we obtain

$$\psi = \frac{1}{8\pi} \int_0^{2\pi} d\phi \ln \{A - C \cos \phi + [Q_0(\phi)]^{1/2}\},$$

(38)

where

$$Q_0(\phi) = (\Omega_4 \Omega_4 + \Omega_2 \Omega_2) \cos \phi - \frac{1}{2} (\Omega_1^2 - \Omega_2^2 - \Omega_3^2 + \Omega_4^2)^2 + 4\Omega_2 \Omega_4 \Omega_2 \Omega_4 \sin^2 \phi \geq 0.$$  

(39)

The analytic properties of $\psi$ will now depend on whether $Q_0(\phi)$ is a complete square.

(a) $Q_0(\phi)$ is a complete square. The only possibility is

$$\Omega_1 \Omega_2 \Omega_3 \Omega_4 = 0,$$

(40)

which can be realized by taking, e.g., $\Omega_1 = \Omega_3 = \Omega_4 = 0$ (since $\Omega_1, \Omega_2, \Omega_3$ and $\Omega_4$ are now single Boltzmann factors). Then

$$\psi = \frac{1}{8\pi} \int_0^{2\pi} d\phi \ln \max\{\Omega_1^2 + \Omega_2^2 - 2\Omega_1 \Omega_4 \cos \phi, \Omega_2^2 + \Omega_3^2 + 2\Omega_2 \Omega_4 \cos \phi\},$$

(41)

Now $\Omega_1, \Omega_2, \Omega_3$ and $\Omega_4$ do not form a closed polygon for $T < T_c$, where $T_c$ is given by the critical condition (27). Then one of the two factors inside the curly brackets in Eq. (41) is always bigger and one finds

$$\psi = \frac{1}{8\pi} \ln \max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}, \quad T < T_c.$$  

(42)

In other words, the system is in a frozen state below $T_c$. Note that in this case, there exists only one transition.

For $T > T_c$, $\Omega_1, \Omega_2, \Omega_3$ and $\Omega_4$ do form a closed polygon. Then there exists $0 < \phi_1 < \pi$,

$$\cos \phi_1 = (\Omega_1^2 + \Omega_2^2 - \Omega_3^2 - \Omega_4^2)/2(\Omega_1 \Omega_4 + \Omega_2 \Omega_4),$$

(43)

such that the two factors in Eq. (41) are equal at $\phi_1$. One finds

$$\psi = \frac{1}{8\pi} \ln \max\{\Omega_1, \Omega_4\}$$

$$+ \frac{1}{8\pi} \int_{\phi_1}^{\pi} d\phi \ln (\Omega_1^2 + \Omega_2^2 + 2\Omega_2 \Omega_4 \cos \phi) d\phi, \quad T > T_c.$$  

(44)

$\psi$ given by Eq. (44) is analytic in $T > T_c$.

The critical case is when $\Omega_1, \Omega_2, \Omega_3$ and $\Omega_4$ just form a polygon ($T = T_c$) or $\phi_1 = 0, \pi$. To examine the singular behavior of Eq. (44) near $T_c$, we note that the integrand vanishes at $T_c$, so it is necessary to compute the derivatives of $\phi$. Using the same argument as in I, we find the first derivatives of $\psi$ behave as $\phi_1$ near $T_c$. This leads to the critical behavior

$$\psi_{\text{slab}} \sim T^{3/2}, \quad T \to T_c.$$  

(45)

(b) $Q_0(\phi)$ is not a complete square. $Q_0(\phi)$ is not a complete square if and only if

$$\Omega_2 \Omega_4 \Omega_2 \Omega_4 > 0.$$  

(46)

In this case $\psi$ cannot be evaluated in a closed form. It can be shown, however, that the first derivatives of $\psi$ can always be expressed in terms of the complete elliptical integrals of the first and third kinds; consequently, the second derivatives of $\psi$ have a logarithmic divergence. It is instructive that, as we shall now see, the singular behavior of $\psi$ as well as the location of the critical point can be obtained without the recourse of actually carrying out the integration. Such an analysis will be useful in situations, as in (ii) below, when the integrals cannot be evaluated.

Generally a function $\psi(T) given by a double integral of the type (37) is analytic in $T$ unless we can make

$$F_0(\theta, \psi) = 0$$

(47)

at some $T, \theta$. We show in Theorem I of Appendix B that, if $Q_0(\phi)$ is not a complete square, (47) holds only at one of the four points given by (B1). For definiteness consider one of the zeros,

$$\theta = \phi = 0, \quad \Omega_1 = \Omega_2 + \Omega_3 + \Omega_4,$$

(48)

and focus attention on the critical region. To extract the singular part of $\psi$, we expand $F_0(\theta, \psi)$ about $\theta = \phi = 0$

$$\psi_{\text{slab}} \sim \int_0^1 \int_0^1 d\theta d\phi \ln (\Omega_1 - \Omega_2 - \Omega_3 - \Omega_4)^2$$

$$+ \alpha \theta^2 + \beta \phi^2 + \gamma \phi \theta^2,$$

(49)

where

$$\alpha = B - D - E, \quad \beta = 2(D - E), \quad \gamma = C - D - E.$$  

(50)

In Eq. (49), only the lower integration limits are needed, and for a given $T$, the integration is restricted in a region around $\theta = \phi = 0$ such that the quantity inside the square brackets is positive. To evaluate this integral, we rotate the $\{\theta, \phi\}$ coordinate by an angle $\frac{1}{2} \tan^{-1} [\beta/(\alpha - \gamma)]$ to eliminate the $\theta \phi$ term. This leads to

$$\psi_{\text{slab}} \sim \int_0^1 d\theta \int_0^1 d\phi \ln (\Omega_1 - \Omega_2 - \Omega_3 - \Omega_4)^2$$

$$+ \frac{1}{2} (\alpha + \gamma + 5) \theta^2 + \frac{1}{2} (\alpha + \gamma - 5) \phi^2,$$

(51)

where

$$\delta = [\beta^2 + (\alpha - \gamma)^2]^{1/2}.$$  

(52)

Provided that the coefficients of $\theta^2$ and $\phi^2$ in Eq. (51) are not zero, the integral (51) can be meaningfully evaluated. In order to obtain an expansion in $t$, we rewrite Eq. (51) in terms of

$$\tau = \Omega_1 - \Omega_2 - \Omega_3 - \Omega_4,$$

(53)
so that
\[ \alpha + \gamma = (\Omega_2 + \Omega_3)^2 + (\Omega_2 + \Omega_4)^2 + (2 \Omega_2 + \Omega_3 + \Omega_4)^2, \]
\[ \delta^2 - (\alpha + \gamma)^2 = -16 \Omega_3 \Omega_6 \Omega_7 \Omega_8 - 4 (\Omega_2 + \Omega_3) (\Omega_2 + \Omega_4) \times (\Omega_3 + \Omega_4) + 4 (\Omega_2 \Omega_3 + \Omega_2 \Omega_4 + \Omega_3 \Omega_4)\gamma^2. \]

Now as \( T \to T_c \), \( \tau \to 0 \), we see that, to the leading order in \( \tau \), Eq. (51) is of the form
\[ \psi_{\text{size}} \sim \int d\theta \int d\phi \ln (t^2 + \rho^2 + q^2), \]
\[ \psi_{\text{size}} \sim t^{3/2} \ln |t|. \]

This is the result quoted in Eq. (28).\(^{11}\)

If \( \Omega_2 \Omega_3 \Omega_4 \neq 0 \) [\( Q_0 (\phi) \) is a complete square], both \( \rho \) and \( q \) are nonzero and Eq. (55) can be evaluated to give\(^{17}\)
\[ \psi_{\text{size}} \sim t^{3/2} \ln |t|. \]

This is the result quoted in Eq. (28).\(^{11}\)

if \( \Omega_2 \Omega_3 \Omega_4 \neq 0 \) [\( Q_0 (\phi) \) is a complete square], the above argument breaks down because \( q = 0 \) at \( T_c \) [or more precisely \( q = 0 (\tau) \)]. Fortunately this case has been considered in (a) above. A direct way to identify this special case is to observe that \( \Omega_2 \Omega_3 \Omega_4 \Omega_3 = 0 \) is precisely
\[ \beta^2 = 4 \alpha \gamma \] at \( T_c \).

Thus the condition under which (56) breaks down is when the discriminant of the quadratic form \( \alpha \phi^2 + \beta \phi + \gamma \) in Eq. (49) vanishes at \( T_c \).

(i) Models satisfying the free-fermion conditions (25) and (26b)

The free energy is now given by
\[ \psi = \frac{1}{16 \pi} \int_0^{2 \pi} d\theta \int_0^{2 \pi} d\phi \ln F_1 (\theta, \phi), \]
\[ F_1 (\theta, \phi) = F_0 (\theta, \phi) - 4 \Delta \sin^2 \phi. \]

Carrying out the \( \theta \) integration, we obtain
\[ \psi = \frac{1}{8 \pi} \int_0^{2 \pi} d\phi \ln \left[ A - C \cos \phi - 2 \Delta \sin^2 \phi + \{ Q(\phi) \}^2 \right], \]
where
\[ Q(\phi) = [2 \Delta \sin^2 \phi + (\Omega_2 \Omega_4 + \Omega_2 \Omega_3) \cos \phi - \frac{1}{2} (\Omega_2^2 - \Omega_3^2 - \Omega_4^2 + \Omega_5^2)]^2 + 8 \Delta \Omega_3 \Omega_4 (1 - \cos \phi) \sin^2 \phi + 4 \Delta_1 \sin^2 \phi \]
\[ = [2 \Delta \sin^2 \phi - (\Omega_2 \Omega_4 + \Omega_2 \Omega_3) \cos \phi + \frac{1}{2} (\Omega_2^2 - \Omega_3^2 - \Omega_4^2 + \Omega_5^2)]^2 + 8 \Delta \Omega_3 \Omega_4 (1 + \cos \phi) \sin^2 \phi + 4 \Delta_1 \sin^2 \phi \geq 0, \]

with \( \Delta_1, \Delta_2 \) defined in Eq. (29). [The last step in Eq. (60) will be proved in Appendix B.] Again, we consider the following two cases separately.

(a) \( Q(\phi) \) is a complete square. Excluding the case \( \Omega_2 \Omega_3 \Omega_4 \Omega_5 = 0 \) considered in (a) above, \( Q(\phi) \)
given by Eq. (60) can be a complete square only when either
\[ \Omega_2 \Omega_3 = \Delta_1 = 0 \] or \( \Omega_2 \Omega_4 = \Delta_2 = 0 \).

(61)

Only the first case will be considered, as the two are obviously related by symmetry. The first relation in Eq. (61) can be realized by taking, e.g.,
\[ \omega_3 = \omega_4 = \omega_5 = \omega_6 = 0 \]
(62)
or
\[ \Omega_2 = v_1 + v_2, \quad \Omega_3 = v_4, \]
\[ \Omega_3 = 0, \quad \Omega_4 = v_4 + v_5, \quad \Delta = v_1 v_2 = v_2 v_3. \]
(63)

Here the last equality follows from the free-fermion conditions (25) and (26b). Now Eq. (59) becomes
\[ \psi = \frac{1}{8 \pi} \int_0^{2 \pi} d\phi \ln \max \{ \Omega_2^2, f(\phi) \}, \]
(64)
where
\[ f(\phi) = \Omega_2^2 + \Omega_4^2 - 2 \Delta \Omega_3 \Omega_4 \cos \phi - 4 \Delta \sin^2 \phi. \]

For \( T < T_c \), where \( T_c \) is given by Eq. (27), \( \Omega_1, \Omega_2, \Omega_3 \) and \( \Omega_4 \) do not form a triangle. Then one of the two factors inside the curly brackets of Eq. (64) prevails, and the integral can be performed. After some algebra we find
\[ \psi = \frac{1}{8 \pi} \int_0^{2 \pi} d\phi \ln \max \{ \Omega_2^2, f(\phi) \}, \]
(65)
where \( G(v_1, v_2, v_4) = \ln (v_1 v_2) + \ln \max (v_1 v_2, v_2 v_3) \)
\[ + \ln \max (v_2 v_3, v_3 v_1). \]
(66)

Therefore, the system is in a frozen state for \( T < T_c \). Note that with the weights given by Eq. (62) or (63), there can exist only one transition.

For \( T > T_c \), there is a triangular relationship between \( \Omega_1, \Omega_2, \) and \( \Omega_4 (\Omega_5 = 0) \). Since \( f(\phi) \) is monotonic in \( \phi \) in \( \{0, \pi\} \), there exists \( \phi_1 \) such that
\[ \Omega_2^2 = f(\phi_1). \]
(66)

Then
\[ \psi = \frac{1}{8 \pi} \int_0^{2 \pi} d\phi \ln \Omega_2^2 f(\phi) = \frac{1}{8 \pi} \int_0^{2 \pi} d\phi \ln \Omega_2^2 f(\phi), \]
(67)

This expression is of the same form as Eq. (44); hence, following the same argument, we obtain
\[ \psi_{\text{size}} \sim t^{3/2}, \quad T \to T_c+. \]
(68)
Also belonging to this category is the staggered free-fermion ice-rule model considered in I specified by

\[ \begin{align*}
\omega_1 &= \omega'_1 = \omega_2' = \omega_3 = \omega_4 = 0, \\
\omega_3 \omega_4 &= \omega_2' \omega_6', \quad \omega_3' \omega_4' = \omega_2 \omega_6.
\end{align*} \tag{69} \]

Using the present method, we find \( \Delta_2 = \Omega_2 / \Omega_4 = 0 \).

(b) \( Q(\phi) \) is not a complete square. Because of the presence of the \( \sin^4 \phi \) term in \( Q(\phi) \), \( \psi \) and its derivatives cannot be evaluated in closed forms. The method of analysis used in (ib) is now useful.

It is shown in Theorem II of Appendix B that, if \( Q(\phi) \) is not a complete square, the zeros of \( F_1(\theta, \phi) \) are given by \( B(10) \). Consider, e.g., the expansion of \( F_1(\theta, \phi) \) about \( \phi = 0 \) in Eq. (57). This will give us the singular behavior of \( \psi \) if \( \Omega_1 = \Omega_2 + \Omega_3 + \Omega_4 \) is a critical point.

Following the argument in (ib) step by step, with \( \gamma = C - D - E - 4 \Delta > 0 \) and

\[ \delta^2 - (\alpha + \gamma)^2 = -16 \Delta_1 - 4 (\Omega_2 + \Omega_3) (\Omega_2 + \Omega_4) (\Omega_3 + \Omega_4) \]

\[ -4 \Delta_2 + 4 (\Omega_2 + \Omega_3 + \Omega_2' + \Omega_3') \gamma^2, \tag{70} \]

we see that if \( \Delta_2 \neq 0 \) at \( \tau = 0 \), we have \( q = \alpha + \gamma = 0 \) as \( \tau \to 0 \). The same argument used in (ib) now leads to, near the critical point \( \Omega_1 = \Omega_2 + \Omega_3 + \Omega_4, \)

\[ \psi_{sag} \sim t \ln |t|, \quad t \to 0. \tag{71} \]

The argument breaks down if \( \Delta_1 = 0 \). If we also have \( \Omega_2 \Omega_3 = 0 \), as given by Eqs. (62) and (63), then \( Q(\phi) \) is a complete square, and the case has been considered in (iia). If \( \Omega_2 \Omega_3 \neq 0 \), it may be verified that we have either \( \Delta_2 = \Omega_4 = 0 \) so \( Q(\phi) \) is again a complete square, or \( \Omega_1 = \Omega_4 \) so \( \Omega_1 = \Omega_2 + \Omega_3 + \Omega_4 \) is not a critical point, and the expansion about \( \phi = 0 \) is irrelevant.

Similarly, expansions of \( F_1(\theta, \phi) \) about \( \{\theta, \phi\} = \{0, \pi\} \) or \( \{\pi, 0\} \) lead to the singular behavior \( t^2 \ln |t| \) except when \( \Delta_2 = 0 \). In the latter case, we must have \( \Omega_2 \Omega_3 = 0 \) to relate the expansions to the critical points. These cases have been considered in (iia).

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APPENDIX A: PFAFFIAN SOLUTION

Procedures of obtaining the Pfaffian solution for the staggered free-fermion eight-vertex model (18) follow closely that of I. First we write

\[ Z(\omega_i, \omega_i') = \langle \omega_2 \omega_2' \rangle^{1/2} Z(u_i, u_i'), \tag{A1} \]

where \( u_i = \omega_i / \omega_6 \), \( u_i' = \omega_i' / \omega_6' \). \( Z(u_i, u_i') \) is then converted into a dimer generating function \( Z^d \).

To evaluate \( Z^d \), we proceed exactly as in I; the only difference here is that a unit cell of the dimer terminal lattice is now given as shown in Fig. 4. \(^{18} \)

It is easily checked, as in Fig. 6 of I, that this unit cell generates all the required vertex weights.

Following the same procedure, we then obtain

\[ \psi = \frac{1}{4(2\pi)^3} \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \ln \left[ \omega_2^3 \omega_2' \omega_2'' D(\alpha, \beta) \right], \tag{A2} \]

where

\[
D(\alpha, \beta) =
\begin{vmatrix}
0 & u_5' & -u_6 & -u_5 & 0 & 1 & 0 & 0 \\
-u_5' & 0 & u_6 & -u_5 & -e^{i\alpha} & 0 & 0 & 0 \\
u_6 & -u_6 & 0 & u_4 & 0 & 0 & 0 & -e^{i(\alpha-\beta)} \\
u_5' & u_7 & -u_4 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{i\alpha} & 0 & 0 & 0 & u_5' & -u_6' & -u_5' \\
-1 & 0 & 0 & 0 & -u_3' & 0 & u_6' & -u_5' \\
0 & 0 & 0 & e^{i\beta} & u_5' & -u_6' & 0 & u_4' \\
0 & 0 & -e^{i(\alpha-\beta)} & 0 & u_5' & u_4' & -u_5' & 0
\end{vmatrix}. \tag{A3}
\]

After changing the variables \( \alpha = \theta + \phi \), \( \beta = \theta \), Eq. (A2) reduces to Eq. (19) in the text. Note that we can see directly from (A3) that \( D(\alpha, \beta) \) factorizes if \( u_5' = u_6' = u_5'' = 0 \), a result quoted in I.
APPENDIX B: ZEROS OF $F_0$ AND $F_1$

In this appendix, we determine the zeros of $F_0(\theta, \phi)$ and $F_1(\theta, \phi)$, defined, respectively, by Eqs. (21) and (58).

**Theorem I**

$F_0(\theta, \phi) = 0$ for all $\theta$ and $\phi$.

and

$F_0(\theta, \phi) = 0$ if and only if,

(a) for $\Omega_2 \Omega_3 \Omega_4 \neq 0$, at the following points:

\[
\begin{align*}
\theta = \phi = 0, & \quad \Omega_1 = \Omega_2 + \Omega_3 + \Omega_4, \\
\theta = \phi = \pi, & \quad \Omega_2 = \Omega_1 + \Omega_3 + \Omega_4, \\
\theta = 0, & \quad \phi = \pi, \quad \Omega_3 = \Omega_1 + \Omega_2 + \Omega_4, \\
\theta = \pi, & \quad \phi = 0, \quad \Omega_4 = \Omega_1 + \Omega_2 + \Omega_3;
\end{align*}
\]

(B1)

(b) for $\Omega_2 \Omega_3 \Omega_4 \neq 0$, at

\[
\begin{align*}
\cos \theta &= (\Omega_2^2 - \Omega_3^2 + \Omega_3^2 + \Omega_4^2)/(2(\Omega_1 \Omega_2 + \Omega_2 \Omega_3)), \\
\cos \phi &= (\Omega_1^2 - \Omega_3^2 - \Omega_4^2)/(2(\Omega_1 \Omega_4 + \Omega_2 \Omega_3)),
\end{align*}
\]

(B2)

(B3)

Note that Eqs. (B2) and (B3) include Eq. (B1) and have real solutions only when $\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 \geq 2 \max \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$.

**Proof:** We write

\[
F_0(\theta, \phi) = 2a + 2b \cos \theta + 2c \sin \theta > 2[a - (b^2 + c^2)^{1/2}],
\]

(B4)

where

\[
\begin{align*}
a &= \frac{1}{2}(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) - (\Omega_1 \Omega_2 - \Omega_3 \Omega_4) \cos \phi, \\
b &= \Omega_2 \Omega_4 - \Omega_1 \Omega_3 + (\Omega_1 \Omega_4 - \Omega_2 \Omega_3) \cos \phi, \\
c &= (\Omega_1 \Omega_4 - \Omega_2 \Omega_3) \sin \phi,
\end{align*}
\]

and the equal sign in Eq. (B4) holds when

\[
\begin{align*}
\cos \theta &= -b/(b^2 + c^2)^{1/2}, \\
\sin \theta &= -c/(b^2 + c^2)^{1/2}.
\end{align*}
\]

(B6)

It is readily verified that

\[
a^2 - b^2 - c^2 = Q_0(\phi) \geq 0,
\]

(B7)

where $Q_0(\phi)$ has been given in Eq. (39). Since $a \geq 0$, we conclude from Eqs. (B4) and (B7) that

$F_0(\theta, \phi) > 0$.

(B8)

We also see that $F_0(\theta, \phi)$ vanishes if and only if Eq. (B6) holds, and

$Q_0(\phi) = 0$.

(B9)

Using Eq. (39) for $Q_0(\phi)$, Eq. (B9) now leads to Eqs. (B1)–(B3). Q. E. D.

**Theorem II**

$F_1(\theta, \phi) = 0$ if and only if,

(a) for $\Delta \leq 0$, and for $\Omega_2 \Omega_3$ or $\Delta_1 \neq 0$ (or $\Omega_4 \Omega_4$ or $\Delta_2 \neq 0$ by symmetry), and $\Delta > 0$, at one of the following points:

\[
\begin{align*}
\theta = \phi = 0, & \quad \Omega_1 = \Omega_2 + \Omega_3 + \Omega_4, \\
\theta = \phi = \pi, & \quad \Omega_2 = \Omega_1 + \Omega_3 + \Omega_4, \\
\theta = 0, & \quad \phi = \pi, \quad \Omega_3 = \Omega_1 + \Omega_2 + \Omega_4, \\
\theta = \pi, & \quad \phi = 0, \quad \Omega_4 = \Omega_1 + \Omega_2 + \Omega_3;
\end{align*}
\]

(B10)

(b) for $\Omega_2 \Omega_3 = 0$, $\Delta_1 = 0$, $\Delta > 0$, at

\[
2 \Delta \sin^2 \phi + \Omega_1 \Omega_4 \cos \phi - \frac{1}{2}(\Omega_1^2 - \Omega_2^2 - \Omega_3^2 + \Omega_4^2) = 0,
\]

(B11)

and $\phi$ given by Eq. (B6);

(c) for $\Omega_1 \Omega_4 = 0$, $\Delta_2 = 0$, $\Delta > 0$, at

\[
2 \Delta \sin^2 \phi - \Omega_2 \Omega_4 \cos \phi + \frac{1}{2}(\Omega_1^2 - \Omega_2^2 - \Omega_3^2 + \Omega_4^2) = 0,
\]

(B12)

and $\phi$ given by Eq. (B6). Here $\Delta_1$, $\Delta_2$ are defined in Eq. (29). Also note that Eqs. (B11) and (B12) include (B10).

**Proof:** Since

\[
F_1(\theta, \phi) = F_0(\theta, \phi) - 4 \Delta \sin^2 \phi,
\]

(B13)

for $\Delta \leq 0$ the theorem follows directly from Theorem I. Therefore, in the following we assume $\Delta > 0$, unless otherwise noted.

We write

\[
F_1(\theta, \phi) = 2a' + 2b \cos \phi + 2c \sin \phi
\]

\[
> 2[a' - (b^2 + c^2)^{1/2}],
\]

(B14)

where

\[
a' = a - 2 \Delta \sin^2 \phi,
\]

and the equal sign holds when $\theta$ is given by Eq. (B6). It is verified that, for all $\Delta$,

\[
a'^2 - b^2 - c^2 = Q(\phi) \geq 0,
\]

(B15)

where $Q(\phi)$ is given by Eq. (60) and the non-negativity of $Q(\phi)$ will be proved later. Now

\[
a' \geq 0, \quad \text{for all } \phi.
\]

(B16)

This can be seen by noting $a' \geq 0$ at $\phi = 0, \pi$; so the minimum of $a'$, if any, occurs at $\phi = \phi_0$, $\cos \phi_0 = (\Omega_1 \Omega_4 - \Omega_2 \Omega_3)/4 \Delta$. Hence,

\[
(a')_{\text{min}} = A - 2 \Delta (1 + \cos^2 \phi_0) \geq A - 4 \Delta \geq 0,
\]

(B17)

where the last step in Eq. (B17) follows from

\[
A \geq 2(v_1 v_2 + v_2 v_3 + v_3 v_4 + v_4 v_5),
\]

(B18)

and, for $\Delta > 0$,

\[
v_1 v_2 + v_2 v_3 > \Delta, \quad v_3 v_4 + v_4 v_5 > \Delta.
\]

(B19)

From Eqs. (B14)–(B16), we conclude that, for $\Delta > 0$,

\[
F_1(\theta, \phi) > 0,
\]

(B20)
and \( F_1(\theta, \phi) \) vanishes only at
\[
Q(\phi) = 0, \tag{B21}
\]
and \( \theta \) given by Eq. (B6). Using Eq. (60) for \( Q(\phi) \), Eq. (B20) now leads to the remainder (\( \Delta > 0 \)) of Theorem II.

Thus it remains only to show \( Q(\phi) > 0 \).

For \( \Delta < 0 \) this follows from \( Q_{\text{ii}}(\phi) > 0 \). For \( \Delta > 0 \) we use (60) for \( Q(\phi) \); so the proof is completed if we can show that we always have either \( \Delta_1 > 0 \) or \( \Delta_2 > 0 \). In the following we shall use only

\[
\begin{align*}
u = (v_1 v_3 + v_2 v_4)(v_1 v_4 + v_2 v_3) & + (v_1 v_5 + v_6 v_7)(v_5 v_8 + v_6 v_7) = v_1 v_4 (v_1 - v_2)^2 + v_1 v_2 (v_2 - v_3)^2 + v_1 v_4 (v_4 - v_5)^2 + v_1 v_4 \Omega_3^2 \\
v = (v_1 v_3 + v_2 v_4)(v_1 v_4 + v_2 v_3) & + (v_1 v_5 + v_6 v_7)(v_5 v_8 + v_6 v_7) = (v_1 v_3 + v_2 v_4)(v_3 v_4 + v_6 v_7) + (v_1 v_5 + v_6 v_7)(v_5 v_8 + v_6 v_7),
\end{align*}
\]

since
\[
\begin{align*}
u_1 v_4 & + v_2 v_3 - v_1 v_3 - v_2 v_4 - v_1 v_2 v_4 (\omega_1 \omega_4 + \omega_2 \omega_4) = (v_1 - v_2)(v_4 - v_2) + \omega_1 \omega_2 (\omega_1 \omega_4 + \omega_2 \omega_4) \\
& = v_1 (v_1 - v_2) + v_4 (v_4 - v_2) + 2 \omega_1 \omega_2 (\omega_1 \omega_4 + \omega_2 \omega_4) = (v_1 - v_2) + v_4 (v_4 - v_2) + 2 \omega_1 \omega_2 (\omega_1 \omega_4 + \omega_2 \omega_4) \geq 0, \quad \text{for all } v_1, v_2, v_3, v_4.
\end{align*}
\]

We have
\[
v \equiv (v_1 v_2 + v_3 v_4 - \omega_1 \omega_2 (\omega_1 \omega_4 + \omega_2 \omega_4))(v_1 v_3 + v_4 v_5 + v_3 v_4 + v_5 v_6) = 2 \omega_1 \omega_2 (\omega_1 \omega_4 + \omega_2 \omega_4) \Omega_4 \Omega_5.
\]

Combining Eqs. (B23)–(B25), we find
\[
\Delta_1 \geq v_1 v_2 \Omega_4^2 + v_3 v_4 \Omega_5^2 + 2 \omega_1 \omega_2 (\omega_1 \omega_4 - \omega_2 \omega_4) \Omega_4 \Omega_5 - \Delta (\Omega_2 + \Omega_3)^2 = \omega_3 \omega_4 (\omega_1 \omega_4 - \omega_2 \omega_4) (\Omega_2 + \Omega_3)^2 + \omega_1 \omega_2 \omega_3 \omega_4 \Omega_4^2 + \omega_1 \omega_3 \omega_2 \Omega_4 \Omega_5.
\]

Similarly we can show (by symmetry)
\[
\Delta_2 \geq \omega_1 \omega_2 (\omega_1 \omega_4 - \omega_2 \omega_4) (\Omega_1 + \Omega_4)^2.
\]

From Eqs. (B26) and (B27) we see that we have always either \( \Delta_1 > 0 \) or \( \Delta_2 > 0 \). Q. E. D.

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8 Two vertex configurations are conjugate to each other if they are related by a bond-hole interchange.
10 There is a misprint in Fig. 11 of Ref. 1. The symbol \( J' \) and \( J'' \) there should be interchanged.
11 It should be pointed out that if the two transition temperatures determined by (33) coalesce into a single one, we have near this \( T_c \) the \( \Omega_1 - \Omega_2 - \Omega_3 - \Omega_4 \) perturbations determine the single \( T_c \). The singular part of \( \delta \) then behaves as \( \delta \propto \ln |t| \) and the specific heat is finite at \( T_c \). This anomalous behavior, which is reminiscent of that found in a decorated Ising system [H. T. Yeh, Physica 64, 427 (1973)] and occurs only for some special vertex energies related by a pair of transcendental parametric equations, may be disregarded in physical considerations.
14 Discussion here is similar to that following Eq. (39) in I.
15 H. S. Green and C. A. Hurst, Order-Disorder Phenomena, edited by I. Prigogine (Interscience, New York, 1964), Sec. 5.3.
16 The validity of Eq. (47) does not necessarily imply nonanalyticity in \( \delta \), however. Consider, e.g., \( \delta = \phi \), defined by Eq. (49). If \( Q_{\phi} (\phi) \) is a complete square, our analysis shows that Eq. (47) can hold at \( \phi_l \) (and appropriate \( \theta \)) for all \( T > T_c \), while \( \phi_l \) is analytic.
17 For \( p \neq 0 \) change variables by \( \theta = \cos \phi \); \( \delta = \sin \phi \), and change variable by \( r = \cos \phi \) where \( r = \min \{|p|, |q|\} \) and change variable by \( \delta = \phi \).
18 One can also use the planar dimer model introduced in Ref. 6 and arrive at a \( 12 \times 12 \) determinant in Eq. (A2).