On the Eigenvalues of Orbital Angular Momentum

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Using only the elementary commutation relations in quantum mechanics, it is shown that the eigenvalues of \( \hat{L}_z = \hat{x} \hat{p}_y - \hat{y} \hat{p}_x \) are integers.

The eigenvalue problem for the orbital angular momentum operator

\[
\hat{L} = \hat{r} \times \hat{p}
\]  

has been one of the least satisfactorily discussed topics in elementary quantum mechanics. In the discussions found in most of the textbooks, one usually starts from the commutation relations implied by (1) and derives the result that the eigenvalues of \( \hat{L}_z \) can only be half-integers (0, ±1/2, ±1, ±3/2, ...). This result is obtained by purely abstract considerations without any need for the use of function spaces. It is sufficient to simply use an abstract Hilbert space without demanding any specific realization of the space. At this point the problem of the elimination of the \( \frac{1}{2} \) integral eigenvalues arises. This is usually done by going outside the abstract Hilbert space framework and realizing (1) as an operator in a function space. Then with the help of some further restrictions, such as the single-valuedness requirement on the eigenfunction in the Schrödinger representation, one rules out the half-odd integral values (±\( \frac{1}{2} \), ±\( \frac{3}{2} \), ...). The necessity for inclusion of a physical constraint and the explicit use of a particular representation in the discussion of an eigenvalue problem has caused some uneasy feelings among physicists and has been a subject of considerable debate for many years.

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1) See, for example, E. Merzbacher, Quantum Mechanics (John Wiley & Sons, Inc., New York, 1963), pp. 355 and 174.
While the restriction can certainly be formulated in a variety of seemingly harmless statements, it is nevertheless annoying to have the necessity for introducing such conditions. Several years ago Buchdahl and Louck gave independent derivations of the eigenvalues of $L_z$ without using any requirement. While they both recognized the fact that the correct eigenvalues are implied by the particular form (1) of the orbital angular momentum operator, their arguments do not take the most elegant form. Besides being rather lengthy and quite indirect, their derivations involve the use of particular representations for the orbital angular momentum operator. Shortly thereafter, Merzbacher pointed out the connection between the two-dimensional harmonic oscillator and the angular momentum in three dimensions which provides, for the first time, a direct derivation of the correct eigenvalues. However, as far as we know, this proof has never been adopted in any textbook of quantum mechanics, presumably because the ingenious trick involved is not an everyday tool familiar to all students.

We wish to present in this note another proof which seems to us to be more direct and simpler in structure and, therefore, more suitable for classroom presentations. The proof is abstract in structure depending only on the form of the operator $L_z$ (i.e. that it is built in a specific way out of the operators $\mathbf{r}$ and $\mathbf{p}$) and on the fact we are (as in the general angular momentum theory) working in an abstract Hilbert space.

First let us write $L_z$ as

$$L_z = x(p_y - p_x) - y(p_x - p_y),$$

where

$$A = \sqrt{2} (p_x - i x)$$

(2)
The following relations can then be readily established by using the commutation relations between $r$ and $p(=l)$.

\[
[A, A^\dagger] = 1 \quad (3)
\]
\[
[B, B^\dagger] = 1 \quad (4)
\]
\[
[C, C^\dagger] = 2 \quad (5)
\]
\[
[A^\dagger A, B^\dagger B] = 0 \quad (6)
\]
\[
[C^\dagger C, A^\dagger A + B^\dagger B] = 0 \quad (7)
\]

The proof is based on the following results well-known to all students of quantum mechanics(13) which we now present as two lemmas. If $A$ and $B$ are two operators in a Hilbert space, then:

**Lemma 1.** The commutation relation $[A, A^\dagger] = 1$ implies that the eigenvalues of $A^\dagger A$ are $0, \lambda, 2\lambda, 3\lambda, \ldots$.

**Lemma 2.** If $A, B$ commute, then the eigenvalues of $A + B$ (or $A - B$) are some sums (or differences) of the eigenvalues of $A$ and $B$.

From (3), (4) and Lemma 1, the eigenvalues of $A^\dagger A$ and $B^\dagger B$ are $0, 1, 2, \ldots$ Hence by (6) and Lemma 2, the eigenvalues of $A^\dagger A + B^\dagger B$ are $0, 1, 2, \ldots$ Similarly from (5) and Lemma 1, the eigenvalues of $C^\dagger C$ are $0, 2, 4, \ldots$ Hence from (7), Lemma 2, and the eigenvalues of $A^\dagger A + B^\dagger B$ just deduced, the eigenvalues of $L_z$ can only have positive or negative integral values including zero. This completes the proof.(14)

To summarize, we have shown that the operator $L_z$ defined in a Hilbert space has integral eigenvalues only. The proof does not use any additional condition usually needed in the Schrodinger representation.

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(13) Lemma 1 is proved in almost any elementary textbook in quantum mechanics. See, for example, pp. 349-351 of Ref. 1. Lemma 2 follows from the fact that commuting operators have simultaneous eigenvectors.

(14) Technically speaking, our proof only rules out the non-integral eigenvalues. But this is the desired result.